The commutator I want to calculate is

$$[\Psi(r), H_0] = \frac{1}{2m}\int dr' \left[\Psi^\dagger(r')\nabla^2_{r'}\Psi(r'), \Psi(r)\right]$$

So I need the operators

$$\Psi^\dagger(r') = \sum_k e^{-ikr'} a_k^\dagger$$

$$\Psi(r) = \sum_k e^{ikr} a_k$$

$$\nabla^2_{r'}\Psi(r') = \sum_k \nabla^2_{r'} e^{ikr'} a_k$$

I calculate the commutator using the anti-commutators

$$\begin{aligned}\left[\Psi^\dagger(r')\nabla^2_{r'}\Psi(r'), \Psi(r)\right] &= \Psi^\dagger(r')\left\{\nabla^2_{r'}\Psi(r'), \Psi(r)\right\} - \left\{\Psi^\dagger(r'), \Psi(r)\right\}\nabla^2_{r'}\Psi(r') \\ &= -\delta(r-r')\nabla^2_{r'}\Psi(r')\end{aligned}$$

$$\boxed{[\Psi(r), H_0] = -\frac{1}{2m}\int dr'\delta(r-r')\nabla^2_{r'}\Psi(r') = -\frac{1}{2m}\nabla^2_r\Psi(r)}$$

The other commutator is

$$\left[\Psi^\dagger(r), H_0\right] = \frac{1}{2m}\int dr' \left[\Psi^\dagger(r')\nabla^2_{r'}\Psi(r'), \Psi^\dagger(r)\right]$$

Same procedure

$$\begin{aligned}\left[\Psi^\dagger(r')\nabla^2_{r'}\Psi(r'), \Psi^\dagger(r)\right] &= \Psi^\dagger(r')\left\{\nabla^2_{r'}\Psi(r'), \Psi^\dagger(r)\right\} - \left\{\Psi^\dagger(r'), \Psi^\dagger(r)\right\}\nabla^2_{r'}\Psi(r') \\ &= \Psi^\dagger(r')\left\{\nabla^2_{r'}\Psi(r'), \Psi^\dagger(r)\right\} \\ &= \Psi^\dagger(r')\nabla^2_{r'}\delta(r-r')\end{aligned}$$

$$\begin{aligned}\left[\Psi^\dagger(r), H_0\right] &= \frac{1}{2m}\int dr'\Psi^\dagger(r')\nabla^2_{r'}\delta(r-r') \\ &= \frac{1}{2m}\int dr'\Psi^\dagger(r')\left(-\nabla^2_r\right)\delta(r-r') \\ &= -\nabla^2_r\frac{1}{2m}\int dr'\Psi^\dagger(r')\delta(r-r') \\ &= \boxed{-\frac{1}{2m}\nabla^2_r\Psi^\dagger(r)}\end{aligned}$$

I wonder if this is correct?