Proof that Cosmological redshift is equivalent to Doppler red shift

This proof is based on that presented by Jayant N Narlikar in 'Spectral shifts in General Relativity' in the American Journal of Physics October 1994. It corrects the invalidity of Narlikar's proof arising from his use of bases that do not exist. Equation numbers are chosen to match Narlikar's where applicable.

The FLRW metric centred at S gives the following formula for the line element:

$ds^2 = dt^2 – a(t)^2 [dr^2/(1-kr^2) + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]$ [7]

t, r, θ and φ are the 'comoving coordinates'. r is often denoted by χ, but here we use r.
Where we do not explicitly indicate a coordinate system, the co-moving coordinate system will be assumed.
a(t) is the cosmic scale parameter at cosmic time t.
The values of the metric components in this coordinate system are:
$g_{00} = 1$; $g_{11} = -a(t)^2/(1-kr^2)$; $g_{22} = -a(t)^2r^2$; $g_{33} = -a(t)^2r^2\sin^2\theta$ [7a]
All other components (the off-diagonal ones) are zero. We will only need to use the first two of the four nonzero components.
Since the metric is diagonal in the comoving coordinates, we also have the following inverse components:
$g^{00} = 1$; $g^{11} =-(1-kr^2)/a(t)^2$; g^{22} and g^{33} are not used. [7b]

Let S denote the spacetime event of emitting the signal that is seen by the observer at the spacetime event denoted by O.

Let $\mathbf{W}_O$ and $\mathbf{W}_S$ be the four-velocities of the observer and source at O and S respectively, both of which are stationary with respect to the CMBR. By definition $||\mathbf{W}_O|| = 1 = ||\mathbf{W}_S||$ [9a].

We will be using three different reference frames in what follows:

- the FLRW frame centred on S (the “FS frame”)
- the FLRW frame centred on O (the “FO frame”)
- the inertial (Lorentz) frame of the observer at O (the “MO frame”)

For coordinate-dependent items the applicable coordinate system will be denoted by a *pre-subscript* such as in ${}_{FO}W_S$, except that we will omit the pre-subscript for items expressed in the FS frame, as that is the one we will use the most.

Parameterise the null geodesic Λ from S to O by $\lambda:[0,1]\rightarrow\Lambda$.

Let $\hat{\mathbf{W}}_S$ be the result of parallel transporting $\mathbf{W}_S$ along the null geodesic Λ from S to O, and let $\mathbf{W}^*(u)$ be the intermediate transported vector at λ(u).

Along Λ we have, from [7], since θ and φ do not change, and ds=0:
$dr/\sqrt{1-kr^2} = dt/a(t)$ [10]
The sign is positive because r increases with t (recall that r is radial distance from S).

First we need to write the equation for the geodesic Λ in terms of an affine parameter u. So that the geodesic is λ:[0,1]->Λ. To do this we write the following geodesic equation. But first, note that in what follows we do not specify the centre for our FLRW coordinates. Hence equations 11-16a are valid for any FLRW system, provided the value of r appropriate to the given system is used.

$d^2x^i/du^2 + \Gamma^i_{kl}\,(dx^i/du)\,(dx^k/du) = 0$ [11 – see Schutz 6.51]

We calculate the Christoffel symbol's value for i=0 as follows:

$\Gamma^0_{kl} = ½g^{0\beta}\,(g_{\beta k,l} + g_{\beta l,k} – g_{kl,\beta})$ [11a – see Schutz 6.32]
$= ½g^{00}\,(g_{0k,l} + g_{0l,k} – g_{kl,0})$ [since $g^{0\beta} = 0$ unless $\beta = 0$]
$= ½g^{00}\,(2g_{00,l} – g_{kl,0})$
$= -½g_{kl,0}$ [since $g^{00} = g_{00} = 1$, which is constant]

Hence [11] for i=0 becomes:

$0 = d^2t/du^2 – ½g_{kl,0}(dx^k/du)\,(dx^l/du)$
$= d^2t/du^2 – ½g_{00,0}(dt/du)^2 – g_{11,0}(dr/du)^2$ [since $d\theta/du$ and $d\varphi/du$ must be zero]
$= d^2t/du^2 – ½(\partial(a(t)^2/(1\text{-}kr^2))/\partial t)(dr/du)^2$ [since g_{00} is constant at 1]
$= d^2t/du^2 – (dr/du)^2\, a(t)a'(t)/(1\text{-}kr^2)$ [12]
$= d^2t/du^2 – (dr/dt)^2\,(dt/du)^2\, a(t)a'(t)/(1\text{-}kr^2)$
$= d^2t/du^2 – ((1\text{-}kr^2)/a(t)^2)(dt/du)^2\, a(t)a'(t)/(1\text{-}kr^2)$ [by 10]
$= d^2t/du^2 – (dt/du)^2\, a'(t)/a(t)$
$= a(t)\, d((dt/du)/a(t))/du$

Hence, as $a(t)\neq 0$ we have

$0 = d((dt/du)/a(t))/du$, whence:

$(dt/du)/a(t) = A$ for some constant A. [13]

Hence $dr/du = (dr/dt)(dt/du) = Aa(t) \times (\sqrt{1\text{-}kr^2})/a(t)$ [by 10]
$= A\sqrt{1\text{-}kr^2}$

So $(dr/du)/\sqrt{1\text{-}kr^2} = A$ [14]

This enables u to be determined as a function of r (as the light moves from S to O) and hence of t. It is convenient and permissible to set u=0 at S and u=1 at O.

Integrating [13] we obtain:

$A = \int_0^1 A\, du = \int_{t(O)}^{t(S)} dt/a(t)$ [15]

Let the tangent vector to the geodesic Λ at $\lambda(u)$ be **U**(u). In the comoving coordinates this has components
[dt/du, dr/du, $d\theta/du$, $d\varphi/du$]. The last two are zero and the first two are given by [13] and [14], hence the components are:

$\mathbf{U}(u) = [Aa(t), A\sqrt{1\text{-}kr^2}, 0, 0]$ [16a]

and $\mathbf{U}(1) = [Aa_O, A\sqrt{1\text{-}kr_O^2}, 0, 0]$ [16b]

where r_O is the radial comoving coordinate of O. This is the first equation that assumes a specific centre (S) for the FLRW system.

Let the vector $\mathbf{W}_S$ parallel transported from S to $\lambda(u)$ be $\mathbf{W}^*(u)$. Hence $\mathbf{W}^*(1) = \hat{\mathbf{W}}_S$. Then, as parallel transport preserves magnitude and direction, both $||\mathbf{W}^*(u)|| = g(\mathbf{W}^*(u),\mathbf{W}^*(u))$ and $g(\mathbf{W}^*(u),\mathbf{U}(u))$ must be constant over u [17].

Now we must have $W^{*2}(u) = W^{*3}(u) = 0$ because otherwise the parallel transportation establishes a preferred direction in space, circumferential to S, which contradicts the isotropy assumption. [This is a bit hand-wavy. Seek to make it more rigorous]

Hence $||\hat{\mathbf{W}}_S|| = ||\mathbf{W}^*(1)|| = ||\mathbf{W}^*(0)|| = ||\mathbf{W}_S|| = 1$ [by 9a]. [18a]

And $g(\hat{\mathbf{W}}_S, \mathbf{U}(1)) = g(\mathbf{W}^*(1),\mathbf{U}(1)) = g(\mathbf{W}^*(0),\mathbf{U}(0)) = g(\mathbf{W}_S,\mathbf{U}(0))$ [18b]

From [18a] we get $1 = ||\hat{\mathbf{W}}_S|| = g_O(\hat{\mathbf{W}}_S, \hat{\mathbf{W}}_S) = g_{ik}(O)\, \hat{W}^i_S\, \hat{W}^k_S$
$= g_{00}(O)\,(\hat{W}^0_S)^2 + g_{11}(O)\,(\hat{W}^1_S)^2$

[as off-diagonal elements of g are zero everywhere in spacetime [from 7] and $\hat{W}^2_S = \hat{W}^3_S = W^{*2}(1) =$

$W^{*3}(1) = 0.]$

$= (\hat{W}^0{}_S)^2 - (a_O{}^2/(1-kr_O{}^2)) (\hat{W}^1{}_S)^2$

Hence $(\hat{W}^0{}_S)^2 - (a_O{}^2/(1-kr_O{}^2)) (\hat{W}^1{}_S)^2 = 1$ [19]

From [18b] we get $g_O(\hat{\mathbf{W}}_S, \mathbf{U}(1)) = g_S(\mathbf{W}_S, \mathbf{U}(0))$, hence
$g_{ik}(O)\, \hat{W}^i{}_S\, U(1)^k = g_{ik}(S)\, W^i{}_S\, U(0)^k$
The left-hand side is:
$= g_{00}(O)\hat{W}^0{}_S U(1)^0 + g_{11}(O)\hat{W}^1{}_S U(1)^1$ [since $\hat{W}^2{}_S = \hat{W}^3{}_S = 0$]
$= \hat{W}^0{}_S U(1)^0 - (a_O{}^2/(1-kr_O{}^2))\, \hat{W}^1{}_S U(1)^1$
$= \hat{W}^0{}_S A a_O - (A a_O{}^2/(\sqrt{1-kr_O{}^2}))\, \hat{W}^1{}_S$ [by 16a]
$= A(a_O \hat{W}^0{}_S - (a_O{}^2/(\sqrt{1-kr_O{}^2}))\, \hat{W}^1{}_S)$
The right-hand side is (note that, since we are operating in T_SM here, we have to switch to FO, the FLRW basis centred at O, in order for the components to be well-defined):
$= {}_{FO}g_{00}(S)\ {}_{FO}W^0{}_S\ {}_{FO}U(0)^0 + {}_{FO}g_{11}(S)\ {}_{FO}W^1{}_S\ {}_{FO}U(0)^1$ [since $U(0)^2 = U(0)^3 = 0$ because the light ray is radial. This is also a bit hand-wavy]
$= {}_{FO}g_{00}(S)\ {}_{FO}U(0)^0$
[since the FLRW spatial coordinates of S are constant, so $\mathbf{W}_S = [1,0,0,0]$ in the FO frame]
$= {}_{FO}U(0)^0$ [by 7a]
$= A a_S$ [by 16a]

Hence $a_O \hat{W}^0{}_S - (a_O{}^2/(\sqrt{1-kr_O{}^2}))\, \hat{W}^1{}_S = a_S$ [20]

Next we note that the four-velocity of the observer has components [1,0,0,0] in any FLRW frame (because the observer has zero spatial coordinate velocity in that frame) and also in the observer's inertial frame at O. Hence the time basis vectors of the MO and FS frames must be identical: $\mathbf{e}_0(O) = {}_{MO}\mathbf{e}_0$.
Now $\hat{\mathbf{W}}_S = {}_{MO}\hat{W}^i{}_S\ {}_{MO}\mathbf{e}_i = \hat{W}^i{}_S\, \mathbf{e}_i(O) = \hat{W}^0{}_S\, \mathbf{e}_0(O) + \hat{W}^1{}_S\, \mathbf{e}_1(O) = \hat{W}^0{}_S\ {}_{MO}\mathbf{e}_0 + \hat{W}^1{}_S\, \mathbf{e}_1(O)$
Hence, since both bases are orthogonal, we have ${}_{MO}\hat{W}^0{}_S = \hat{W}^0{}_S$ and, by rotating the Lorentz frame appropriately around O, we can without loss of generality choose our basis vector ${}_{MO}\mathbf{e}_1$ so that it aligns with $\mathbf{e}_1(O)$. Then, since $\hat{\mathbf{W}}_S = [\hat{W}^0{}_S, \hat{W}^1{}_S, 0, 0]$ in the FS basis, we can write $\hat{\mathbf{W}}_S = [\gamma, \gamma V, 0, 0]$ in the MO basis, where ${}_{MO}\hat{W}^0{}_S = \gamma = 1/(\sqrt{1-V^2}) = \hat{W}^0{}_S$ and V has the same sign as $\hat{W}^1{}_S$ (because γ is positive and the corresponding basis vectors, ${}_{MO}\mathbf{e}_1$ and $\mathbf{e}_1(O)$, point in the same direction).
Hence $\|\hat{\mathbf{W}}_S\| = g(\hat{\mathbf{W}}_S, \hat{\mathbf{W}}_S) = \gamma^2 - (\gamma V)^2 = \gamma^2 - (a_O{}^2/(1-kr_O{}^2))\, \hat{W}^1{}_S{}^2$
where we calculate the magnitude in the MO and FS bases and equate the results.
Hence $(\gamma V)^2 = (a_O{}^2/(1-kr_O{}^2)) (\hat{W}^1{}_S)^2$
and so $\gamma V = +a_O/(\sqrt{1-kr_O{}^2}))\, \hat{W}^1{}_S$ [21]
where the sign is positive because γ is positive and V and $\hat{W}^1{}_S$ have the same sign.

Now the red shift is given by:
$1+z = a_O/a_S$ [8]
$= a_O/(a_O \hat{W}^0{}_S - (a_O{}^2/(\sqrt{1-kr_O{}^2}))\, \hat{W}^1{}_S)$ [by 20]
$= 1/(\gamma - (a_O/(\sqrt{1-kr_O{}^2}))\, \hat{W}^1{}_S) = 1/(\gamma - \gamma V)$ [by 21]
$= (1/\gamma)/(1-V) = \sqrt{(1-V^2)}/(1-V) = \sqrt{((1-V)(1+V)/(1-V)^2)} = \sqrt{((1+V)/(1-V))}$ [22]
This is the same formula as for a Doppler shift within a Lorentz frame.