Lorentz Covariance in the Coulomb Gauge

In QED, the Coulomb gauge is specified by $\partial_i A^i = 0, A^0 = 0$ for A^μ the Maxwell potential. Under a Lorentz transformation Λ with infinitesimal parameter ϵ,

$$U(\epsilon)A^\mu(x)U^{-1}(\epsilon) = A^\mu(x') - \epsilon^{\mu\nu}A_\nu(x') + \frac{\partial}{\partial x'_\mu}\lambda(x', \epsilon)$$

for some operator gauge function $\lambda(x', \epsilon)$.

To see this, we first note that for $A(x)$ a solution of $\partial_\mu F^{\mu\nu}$, there exists $A'(x')$ a solution to the Lorentz transformed system. However, A^μ does not quite transform as a 4-vector under a Lorentz transformation and we must also consider a supplementary gauge term which admits the $\partial'^\mu\lambda(x', \epsilon)$.

$A^\mu(x)$ is an operator in some Hilbert space $\mathcal{H}$ with a unitary representation U of the Lorentz group. Thus,

$$A^\mu(x) \xrightarrow{\Lambda} A'^\mu(x') = U(\epsilon)A^\mu(x)U^{-1}(\epsilon).$$

So,

$$\begin{aligned} A \to& A' = (1-\epsilon)A + [\text{gauge terms}], \\ A^\mu \to& A'^\mu(x') = A^\mu(x) - \epsilon^\mu{}_\nu A^\nu(x) + \partial^\mu\lambda(x, \epsilon) \end{aligned}$$

and similarly,

$$x \to x' = (1+\epsilon)x. \qquad (B\&D\ 11.51)$$

Note that by Lorentz invariance of the metric $g^{\mu\nu}$, $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ is an antisymmetric tensor. This is because since we demand invariance, $g = g' \Rightarrow \epsilon g = 0$ necessarily. This determines the antisymmetric form of ϵ from the symmetry of g.

Any field ϕ transforms under a Lorentz transformation Λ as

$$\phi(x) \to \phi'(x') = \rho(\Lambda)\phi(\Lambda^{-1}x).$$

We are therefore interested in the inverse transformation for x. Note that

$$
\begin{aligned}
(1-\epsilon)^{-1} = \sum_{k=0}^{\infty} \epsilon^k &= 1 + \epsilon + \epsilon^2 + \cdots \\
&= 1 + \epsilon + \mathcal{O}(\epsilon^2) \approx 1 + \epsilon. \\
\therefore\ x &= (1+\epsilon)x'.
\end{aligned}
$$

In component form, $x^\kappa = x'^\kappa + \epsilon^\kappa{}_\sigma x'^\sigma$.

As $A^0 = 0$, its Lorentz transformation must also be zero. (Impossible to boost into a non-zero reference frame, etc.) Taking this transformation,

$$
A'^0(x') = A^0(x) - \epsilon^0{}_\nu A^\nu(x) + \partial^0\lambda(x,\epsilon) \stackrel{!}{=} 0.
$$

This implies

$$
\begin{aligned}
\partial^0\lambda &= \epsilon^0{}_\nu A^\nu \\
&= \epsilon^0{}_i A^i.
\end{aligned}
$$

Looking at the Lorentz transformation of $\partial_i A^i$,

$$
\partial'_i A'^i(x') = \partial'_i \Big[A^i(x) - \epsilon^i{}_\nu A^\nu(x) + \partial^i \lambda \Big]. \tag{$\heartsuit$}
$$

Recall that $\frac{\mathrm{d}}{\mathrm{d}x'^i} = \frac{\mathrm{d}}{\mathrm{d}x^\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}x'^i}$, so we must find the Jacobian terms.

$$
\begin{aligned}
\frac{\partial}{\partial x'^i}\, x^\kappa &= \frac{\partial}{\partial x'^i}\left[x'^\kappa + \epsilon^\kappa{}_\sigma x'^\sigma \right] \\
&= \frac{\partial x'^\kappa}{\partial x'^i} + \epsilon^\kappa{}_\sigma \frac{\partial x'^\sigma}{\partial x'^i} \\
&= \delta^\kappa_i + \epsilon^\kappa{}_\sigma \delta^\sigma_i.
\end{aligned}
$$

We can now write $\heartsuit$ as

$$
\begin{aligned}
\partial'_i A'^i(x') &= \partial_\kappa \Big[A^i(x) - \epsilon^i{}_\nu A^\nu(x) + \partial^i\lambda \Big] \Big(\delta^\kappa_i + \epsilon^\kappa{}_\sigma \delta^\sigma_i \Big) \\
&= \delta^\kappa_i \partial_\kappa A^i(x) - \delta^\kappa_i \epsilon^i{}_\nu \partial_\kappa A^\nu(x) + \delta^\kappa_i \partial_\kappa \partial^i \lambda \\
&\quad + \epsilon^\kappa{}_\sigma \delta^\sigma_i \partial_\kappa A^i(x) - \mathcal{O}(\epsilon^2) + \epsilon^\kappa{}_\sigma \delta^\sigma_i \partial_\kappa \partial^i \lambda \\
&= \partial_i A^i(x) - \epsilon^i{}_\nu \partial_i A^\nu(x) + \partial_i \partial^i \lambda + \epsilon^\kappa{}_i \partial_\kappa A^i(x) + \epsilon^\kappa{}_i \partial_\kappa \partial^i \lambda.
\end{aligned}
$$

Introducing some awful notation, we'll define the difference between the Lorentz transformed $\partial_i A^i$ and the original by

$$
\begin{aligned}
\Delta_{\partial A} &\equiv \partial'_i A'^i(x') - \partial_i A^i(x) \\
&= -\epsilon^i{}_\nu \partial_i A^\nu(x) + \underbrace{\partial_i \partial^i \lambda}_{-\triangle\lambda} + \epsilon^\kappa{}_i \partial_\kappa A^i(x) + \underbrace{\epsilon^\kappa{}_i \partial_\kappa \partial^i \lambda}_{\mathcal{O}(\epsilon^2)} \\
&= -\epsilon^i{}_\nu \partial_i A^\nu(x) + \epsilon^\kappa{}_i \partial_\kappa A^i(x) - \triangle\lambda \\
&= \epsilon^\kappa{}_i \partial_\kappa A^i(x) - \epsilon^i{}_\nu \partial_i A^\nu(x) - \triangle\lambda
\end{aligned}
$$

by the definition of the Laplacian $\triangle \equiv -\partial_i \partial^i$. Decomposing κ, ν into space and time,

$$
\begin{aligned}
&= \epsilon^0{}_i \partial_0 A^i(x) + \epsilon^j{}_i \partial_j A^i(x) - \underbrace{\epsilon^i{}_0 \partial_i A^0(x)}_{0} - \epsilon^i{}_j \partial_i A^j(x) - \triangle\lambda \\
&= \epsilon^0{}_i \partial_0 A^i(x) + \underbrace{\epsilon^j{}_i \partial_j A^i(x) - \epsilon^i{}_j \partial_i A^j(x)}_{\reflectbox{∂}} - \triangle\lambda.
\end{aligned}
$$

Examining the $\reflectbox{$\partial$}$-term,

$$
\begin{aligned}
\reflectbox{∂} &= \epsilon^{ji} \partial_j A_i - \epsilon^{ij} \partial_i A_j \\
&= \epsilon^{ji} \Big(\partial_j A_i + \partial_i A_j \Big) = 0
\end{aligned}
$$

by symmetry. As $\Delta_{\partial A} = 0$, this implies

$$
\epsilon^0{}_i \partial_0 A^i = \triangle\lambda. \tag{🍃}
$$

Choose $\triangle\lambda(x, \epsilon) = \phi(x, \epsilon)$. Then $\phi = \epsilon^0{}_i \partial_0 A^i(x)$ and

$$
\lambda = \int \mathrm{d}^3\bar{x}\; G(\mathbf{x} - \bar{\mathbf{x}}) \phi(\bar{\mathbf{x}})
$$

where G is the Green function such that $\triangle G(\mathbf{x}) \equiv \delta^{(3)}(\mathbf{x})$. Several Fourier transformations later (cf. Nigel Buttimore's CED), we arrive at the rather Coulombic result

$$
G(\mathbf{x}) = -\frac{1}{4\pi r}, \qquad r = |\mathbf{x}| = \sqrt{\mathbf{x}^2}
$$

and G is independent of t. By 🍃,

$$
\lambda(t, \mathbf{x}) = -\int \mathrm{d}^3\bar{x}\, \frac{\epsilon^0{}_i \partial_0 A^i(t, \bar{\mathbf{x}})}{4\pi |\mathbf{x} - \bar{\mathbf{x}}|}.
$$

⅃ Corrections to `fionnf@maths.tcd.ie`.