Transient Heat Conduction in 1D Spherical Geometry by Crank-Nicolson Finite Difference

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Consider the 1D spherical, transient heat conduction equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial r^2} + \frac{2\alpha}{r}\frac{\partial T}{\partial r}$$

Where α is the thermal diffusivity $\alpha = {}^{\lambda}/_{\rho c_p}$. For time discretization, use Crank-Nicolson:

$$\frac{\partial T}{\partial t} \approx \frac{T^{n+1} - T^n}{\Delta t} = \frac{1}{2}\left[\alpha\left(\frac{\partial^2 T^{n+1}}{\partial r^2}\right) + \frac{2\alpha}{r}\left(\frac{\partial T^{n+1}}{\partial r}\right) + \alpha\left(\frac{\partial^2 T^{n}}{\partial r^2}\right) + \frac{2\alpha}{r}\left(\frac{\partial T^{n}}{\partial r}\right)\right]$$

For spatial discretization, use central differences to ensure 2nd order accuracy

$$\frac{\partial T}{\partial r} \approx \frac{T_{k+1} - T_{k-1}}{2\Delta r}$$

and

$$\frac{\partial^2 T}{\partial r^2} \approx \frac{T_{k+1} - 2T_k + T_{k-1}}{(\Delta r)^2}$$

Now put them together.

$$\frac{T_k^{n+1} - T_k^n}{\Delta t} = \frac{1}{2}\left[\alpha\left(\frac{T_{k+1}^{n+1} - 2T_k^{n+1} + T_{k-1}^{n+1}}{(\Delta r)^2}\right) + \frac{2\alpha}{r}\left(\frac{T_{k+1}^{n+1} - T_{k-1}^{n+1}}{2\Delta r}\right) + \alpha\left(\frac{T_{k+1}^{n} - 2T_k^{n} + T_{k-1}^{n}}{(\Delta r)^2}\right) + \frac{2\alpha}{r}\left(\frac{T_{k+1}^{n} - T_{k-1}^{n}}{2\Delta r}\right)\right]$$

Get all of the unknown temperature values at the forward time step on one side

$$\frac{T_k^{n+1}}{\Delta t} - \frac{\alpha}{2}\left(\frac{T_{k+1}^{n+1} - 2T_k^{n+1} + T_{k-1}^{n+1}}{(\Delta r)^2}\right) - \frac{\alpha}{r}\left(\frac{T_{k+1}^{n+1} - T_{k-1}^{n+1}}{2\Delta r}\right) = \frac{T_k^n}{\Delta t} + \frac{\alpha}{2}\left(\frac{T_{k+1}^{n} - 2T_k^{n} + T_{k-1}^{n}}{(\Delta r)^2}\right) + \frac{\alpha}{r}\left(\frac{T_{k+1}^{n} - T_{k-1}^{n}}{2\Delta r}\right)$$

This can be written more succinctly as

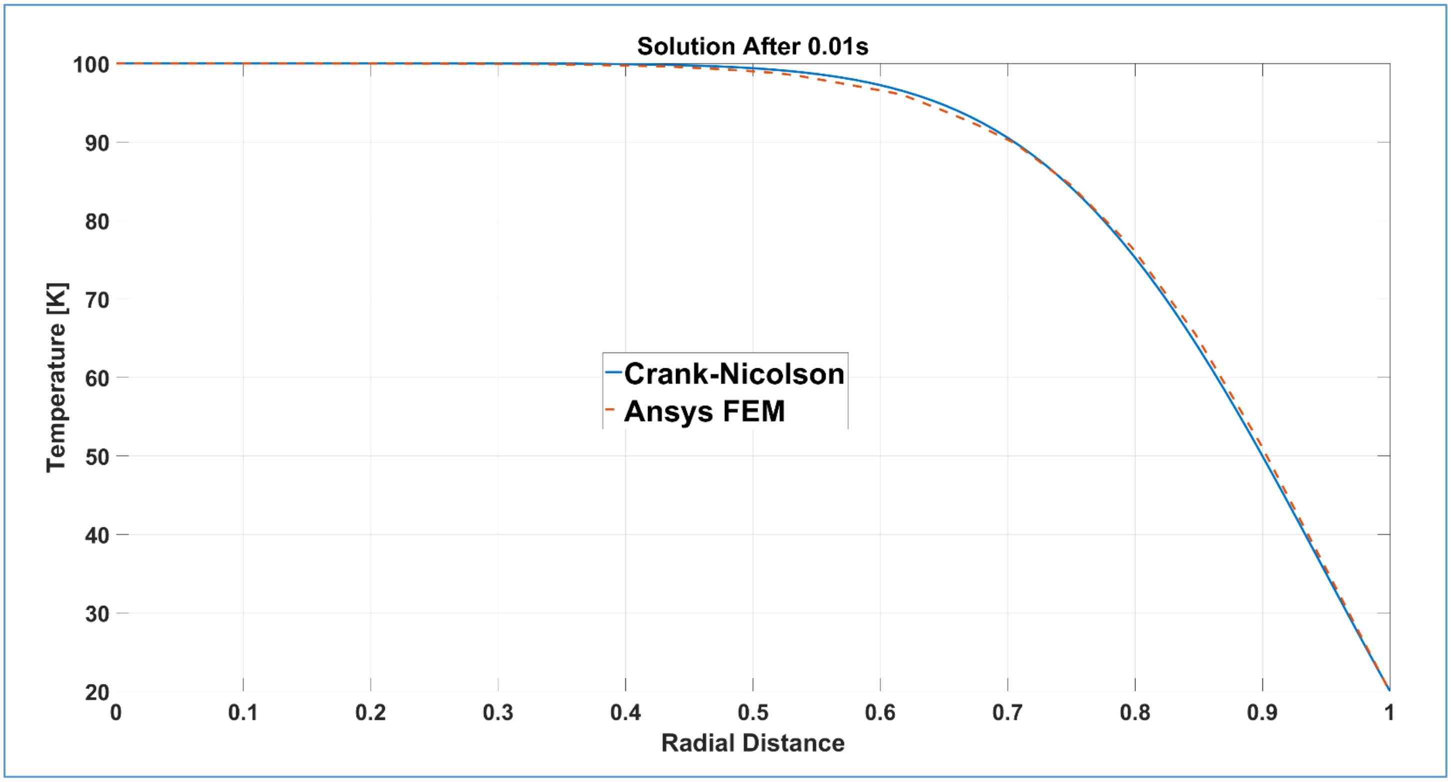
$$\begin{aligned}\left(1 + \frac{\alpha\Delta t}{(\Delta r)^2}\right)T_k^{n+1} - \left(\frac{\alpha\Delta t}{2(\Delta r)^2} + \frac{\alpha\Delta t}{r2\Delta r}\right)T_{k+1}^{n+1} - \left(\frac{\alpha\Delta t}{2(\Delta r)^2} - \frac{\alpha\Delta t}{r2\Delta r}\right)T_{k-1}^{n+1} \\ = \left(1 - \frac{\alpha\Delta t}{(\Delta r)^2}\right)T_k^{n} + \left(\frac{\alpha\Delta t}{2(\Delta r)^2} + \frac{\alpha\Delta t}{r2\Delta r}\right)T_{k+1}^{n} + \left(\frac{\alpha\Delta t}{2(\Delta r)^2} - \frac{\alpha\Delta t}{r2\Delta r}\right)T_{k-1}^{n}\end{aligned}$$

Now use the following symbols to simplify

$$A = \frac{\alpha\Delta t}{(\Delta r)^2}, B = \frac{\alpha\Delta t}{2\Delta r}$$

$$(1 + A)T_k^{n+1} - \left(\frac{A}{2} + \frac{B}{r}\right)T_{k+1}^{n+1} - \left(\frac{A}{2} - \frac{B}{r}\right)T_{k-1}^{n+1} = (1 - A)T_k^n + \left(\frac{A}{2} + \frac{B}{r}\right)T_{k+1}^{n} + \left(\frac{A}{2} - \frac{B}{r}\right)T_{k-1}^{n}$$

If temperatures are prescribed at the center and on the boundary, then we are done. Below is an example. With initial temperature 100K, suddenly the boundary is set to 20K while the center of a sphere with radius 1 m is kept at 100 K. The solution is given after 0.01s for an α=1.5 m^2/s. The C-N solution is compared with a coarse mesh FEM solution. C-N solution used 101 spatial nodes and 30001 time steps, while the FEM used 16 elements and 100 time steps.



If flux symmetry at the center of the sphere is instead enforced, then we use (by L'Hopital at r=0):

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial r^2} + \frac{2\alpha}{r}\frac{\partial T}{\partial r} = 3\alpha \frac{\partial^2 T}{\partial r^2}$$

With Crank-Nicolson

$$\frac{T^{n+1} - T^n}{\Delta t} = \frac{3\alpha}{2}\left[\frac{\partial^2 T^{n+1}}{\partial r^2} + \frac{\partial^2 T^n}{\partial r^2}\right] \approx \frac{3\alpha}{2}\left[\frac{T_{k+1}^{n+1} - 2T_k^{n+1} + T_{k-1}^{n+1}}{(\Delta r)^2} + \frac{T_{k+1}^n - 2T_k^n + T_{k-1}^n}{(\Delta r)^2}\right]$$

Putting like terms on each side of the equality

$$T_k^{n+1} - T_k^n = \frac{3\alpha\Delta t}{2}\left[\frac{T_{k+1}^{n+1} - 2T_k^{n+1} + T_{k-1}^{n+1}}{(\Delta r)^2} + \frac{T_{k+1}^n - 2T_k^n + T_{k-1}^n}{(\Delta r)^2}\right]$$

or more simply

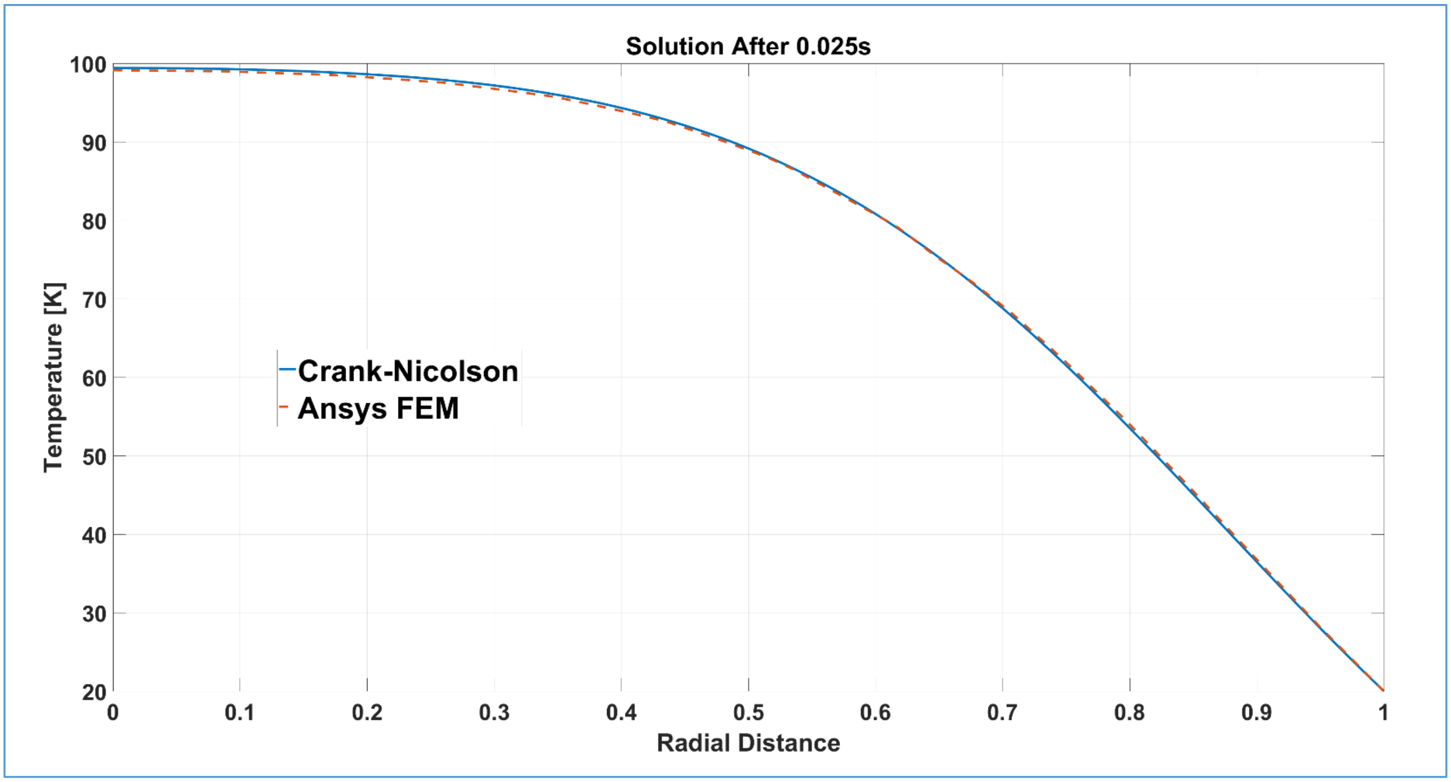
$$T_k^{n+1} - \frac{3\alpha\Delta t}{2(\Delta r)^2}T_{k+1}^{n+1} + \frac{3\alpha\Delta t}{(\Delta r)^2}T_k^{n+1} - \frac{3\alpha\Delta t}{2(\Delta r)^2}T_{k-1}^{n+1} = T_k^n + \frac{3\alpha\Delta t}{2(\Delta r)^2}T_{k+1}^n - \frac{3\alpha\Delta t}{(\Delta r)^2}T_k^n + \frac{3\alpha\Delta t}{2(\Delta r)^2}T_{k-1}^n$$

Now equating the nodes on either side of the center by symmetry

$$T_k^{n+1} - \frac{3\alpha\Delta t}{(\Delta r)^2}T_{k+1}^{n+1} + \frac{3\alpha\Delta t}{(\Delta r)^2}T_k^{n+1} = T_k^n + \frac{3\alpha\Delta t}{(\Delta r)^2}T_{k+1}^n - \frac{3\alpha\Delta t}{(\Delta r)^2}T_k^n$$

$$(1 + 3A)T_0^{n+1} - 3AT_1^{n+1} = (1 - 3A)T_0^n + 3AT_1^n$$

An example was compared to a coarse mesh FEM analysis with the same physical parameters as above but with the flux symmetry at the center enforced. The solution was compared at t=0.025s, as shown below.



If the flux at the boundary is prescribed such that

$$\lambda \frac{\partial T}{\partial r} = q$$

We can use the same 2nd order scheme as before

$$\frac{T_{k+1} - T_{k-1}}{2\Delta r} = \frac{q}{\lambda} \rightarrow T_{k+1} = \frac{2\Delta r q}{\lambda} + T_{k-1}$$

Then we can just plug this in to the original C-N formulation. i.e.,

$$(1 + A)T_k^{n+1} - \left(\frac{A}{2} + \frac{B}{r}\right)T_{k+1}^{n+1} - \left(\frac{A}{2} - \frac{B}{r}\right)T_{k-1}^{n+1} = (1 - A)T_k^n + \left(\frac{A}{2} + \frac{B}{r}\right)T_{k+1}^n + \left(\frac{A}{2} - \frac{B}{r}\right)T_{k-1}^n$$

This gives

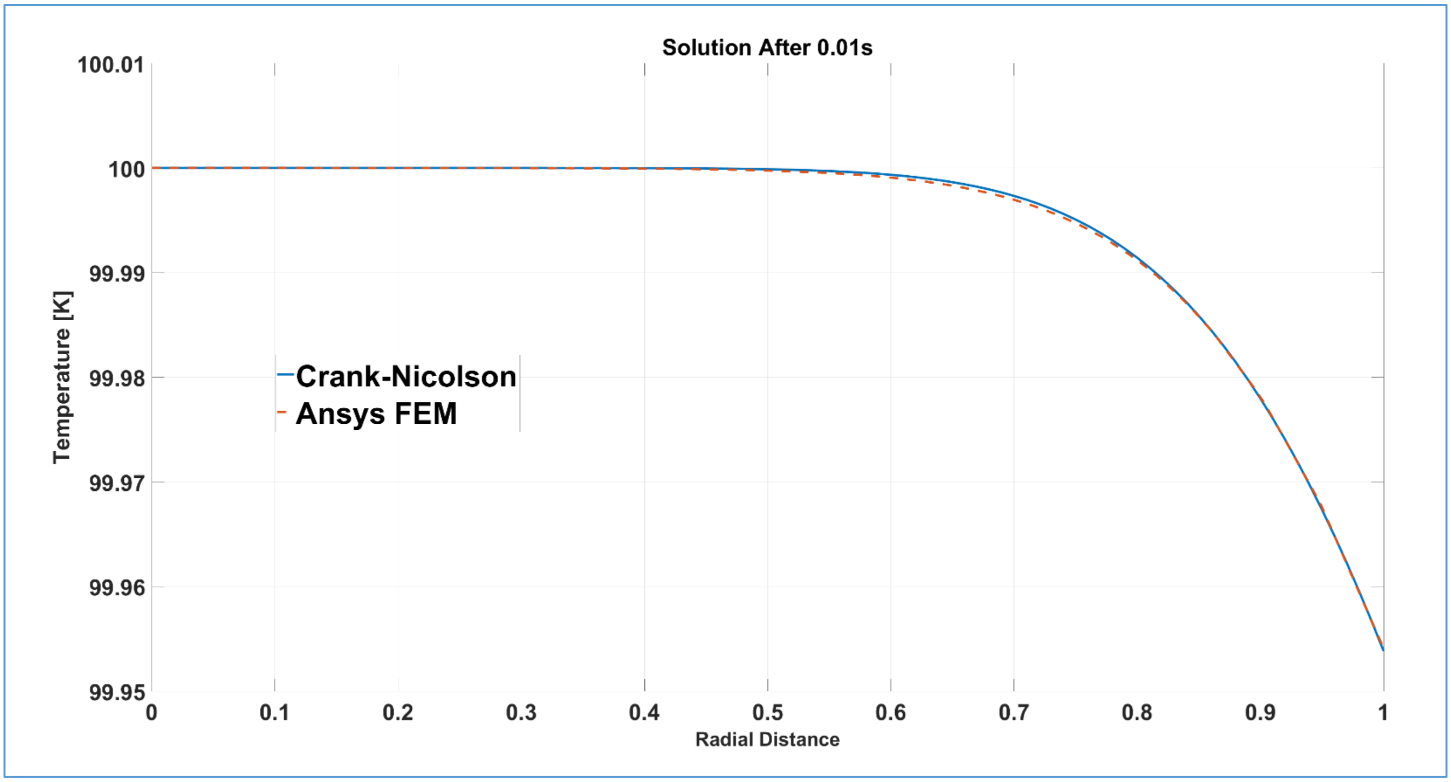
$$(1 + A)T_k^{n+1} - \left(\frac{A}{2} + \frac{B}{r}\right)\left(\frac{2\Delta r q}{\lambda} + T_{k-1}^{n+1}\right) - \left(\frac{A}{2} - \frac{B}{r}\right)T_{k-1}^{n+1} = (1 - A)T_k^n + \left(\frac{A}{2} + \frac{B}{r}\right)\left(\frac{2\Delta r q}{\lambda} + T_{k-1}^n\right) + \left(\frac{A}{2} - \frac{B}{r}\right)T_{k-1}^n$$

Simplified, this reads

$$(1 + A)T_k^{n+1} - \left(\frac{A}{2} + \frac{B}{r}\right)\frac{2\Delta r q}{\lambda} - AT_{k-1}^{n+1} = (1 - A)T_k^n + \left(\frac{A}{2} + \frac{B}{r}\right)\frac{2\Delta r q}{\lambda} + AT_{k-1}^n$$

$$(1 + A)T_K^{n+1} - AT_{K-1}^{n+1} = (1 - A)T_K^n + \left(A + \frac{B}{r}\right)\frac{2\Delta r q}{\lambda} + AT_{K-1}^n$$

A comparison was made with the same physical parameters as above. The center of the sphere has the symmetric flux formulation and the boundary has a negative flux of -.45W/m^2 and thermal conductivity of 1.5 W/mK.



The MATLAB code for the first example is listed below.

```
% Crank-Nicolson in spherical coordinates for fixed center and boundary
% temperature.  Spherical symmetry is assumed.
Te = 1; % End time
Re = 1; % End length
N = 30001; % Number of time nodes (>1)
K = 101; % Number of space nodes (>1)
r = linspace(0,Re,K);  % Nodes
alph = 1.5;  % Diffusivity
dt = Te/(N-1);
dr = Re/(K-1);
A = alph*dt/(dr^2); % Useful parameters
B = alph*dt/(2*dr);
T = 100*ones(K,1); % Initial Temparature
Tc = 100; % Center temperature
Tb = 20;  % Boundary temperature
D0 = [1,repmat(1+A,1,K-2),1];
Dn1 = [-(A/2 - B./r(2:K-1)),0];
Dp1 = [0,-(A/2 + B./r(2:K-1))];
M = diag(D0) + diag(Dn1,-1) + diag(Dp1,1);  % The LHS array

for ii = 1:300
    Tmp = [Tc;
           (A/2 - B./r(2:K-1)').*T(1:K-2) + (1-A)*T(2:K-1) + (A/2 + B./r(2:K-1)').*T(3:K);
           Tb];
    T = M\Tmp;
end
```