EPR, density matrices, and FTL signalling.

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1 The density matrix.

The density operator of a pure state $|\psi\rangle$ is given (by definition) by:

$$\rho_\psi = |\psi\rangle\langle\psi| \tag{1}$$

In a specific basis, this density operator is represented by a matrix, the density matrix. In the case that $|\psi\rangle$ is a basis vector $|e_i\rangle$, the density matrix ρ reduces to a diagonal matrix with only one diagonal element different from zero, namely the i-th element. In the more general case:

$$|\psi\rangle = \sum_{i=1}^{N} \psi_i |e_i\rangle \tag{2}$$

the associated density matrix takes on the form:

$$\rho_{i,j} = \psi_i (\psi_j)^* \tag{3}$$

In standard quantum theory, when a measurement is performed on a state $|\psi\rangle$, corresponding to an operator A, with eigenbasis $|a_i\rangle$, then one should write $\psi\rangle$ in this eigenbasis:

$$|\psi >= \alpha_i |a_i\rangle \tag{4}$$

and after measurement, the system is in the state $|a_i\rangle$ with probability $|\alpha_i|^2$. The fact that the state of the system changed stochastically to one of the eigenstates is called the projection postulate, and the fact that the probability associated with such a projection is given by $|\alpha_i|^2$ is called the Born (probability) rule.

There's another way of seeing what happened during this measurement process. One can say that the pure state $|\psi\rangle$ of which we knew exactly the quantum state, is now a statistical mixture of states $|a_i\rangle$: we lost knowledge of the pure state of the system (and that knowledge is present in the data of the outcome of the measurement).

Now, a density matrix can be used to describe statistical mixtures in the following way: if a statistical mixture is made up of (pure)states $|y_i\rangle$ with weight u_i in a mixture, then the density matrix of the mixture is defined as:

$$\rho_{mix} = \sum_{k=1}^{M} u_k \rho_{y_k} \tag{5}$$

with ρ_{y_k} the density matrix of the pure state $|y_k\rangle$.

Up to now, density matrices only seem to be things that we defined, without any specific meaning. But the meaning of a density matrix comes about when we look at the expectation values of measurements. Consider an (hermitean) operator A describing a measurement. The expectation value of the measured value on a pure state $|\psi\rangle$ is given by:

$$\langle A\rangle = \sum_{i=1}^{N} |\alpha_i|^2 a_i = \langle\psi|A|\psi\rangle \tag{6}$$

This can also be calculated as (trace is invariant under cyclic rotation, and trace of a number is the number):

$$\langle A\rangle = Tr\left(\langle\psi|A|\psi\rangle\right) = Tr\left(|\psi\rangle\langle\psi|A\right) = Tr\left(\rho_\psi A\right) \tag{7}$$

Now, it is important to realize that ALL experimental quantities (including probabilistic frequencies) are expectation values of some or other operator A. Indeed, the probability to satisfy a certain property is simply the expectation value of the projector operator P_i, which projects upon the space of states which have the required property. P_i is a hermitean operator, and can serve as the measurement operator for the said probability.

So this means that the density matrix ρ describes — in the case of a pure state — completely all possible experimental quantities one can extract from an experiment, because it is given by $Tr(\rho A)$. Now, what about mixtures ? The expectation value of a mixture is of course the weighted composition of the expectation values of the individual members of the mixture:

$$\langle A\rangle = \sum_{k=1}^{M} u_k \langle A\rangle_{y_k} \tag{8}$$

where $\langle A\rangle_{y_k}$ is the expectation value of the operator A for the pure state (member of the mixture) $|y_k\rangle$ with weight u_k. Filling this in, we find:

$$\langle A\rangle = \sum_{k=1}^{M} u_k Tr\left(\rho_{y_k} A\right) = Tr\left(\sum_{k=1}^{M} u_k \rho_{y_k} A\right) = Tr\left(\rho_{mix} A\right) \tag{9}$$

So, we see that with our definition of ρ_{mix}, the formula for the expectation value also applies in the case of a mixture! Moreover, as the Trace notation is basis-independent, we can now consider the density matrix more as a basis-independent density operator.

And now we come to a full circle: we saw that the measurement process changed the pure state $|\psi\rangle$ into a mixture of pure states. This can be described by taking the density matrix of the pure state, $\rho_{|\psi\rangle}$ and transforming it into the density matrix of the mixture. This goes as follows: write the density matrix out in the eigenbasis of the measurement that is to be performed, and keep the elements on the diagonal (which are nothing else but the $|\alpha_i|^2$), while putting all the off-diagonal elements to 0. Clearly, this procedure is dependent on the basis in which it is performed! It should be performed in the eigenbasis of the measurement operator A.

2 The reduced density matrix

Consider next a system containing two clearly defined subsystems (for instance, two spatially separated particles). The state space of the first subsystem, H_1, is spanned by a basis $|e_i\rangle$, while the state space of the second subsystem, H_2, is spanned by a basis $|f_j\rangle$. The state space of the entire system is then the tensor product of both: $H = H_1 \otimes H_2$, which is spanned by the basis $|e_i\rangle|f_j\rangle$.

Now, consider a general (pure) state:

$$|\psi\rangle = \sum_{ij} a_{ij}|e_i\rangle|f_j\rangle \tag{10}$$

and let us imagine that we want to find the expectation value of an operator A_1 who is in fact $A_1 \otimes 1$, as A_1 acts only on the first system. Let us suppose that $|e_i\rangle$ is an eigenbasis of A_1 ; we can complete it with just any basis in H_2 because any basis is an eigenbasis of 1.

As such, the expectation value of A_1 for this state becomes:

$$\langle A_1\rangle = \sum_{ij} |a_{ij}|^2\alpha_i \tag{11}$$

The overall density matrix of this state is given by:

$$\rho_{ij,kl} = a_{ij}(a_{kl})^* \tag{12}$$

Let us define the reduced density matrix ρ_1:

$$(\rho_1)_{ik} = \sum_j a_{ij}(a_{kj})^* \tag{13}$$

Note that this sum over j is the 'partial trace' over the H_2 quantities, for a given i and k. Indeed, for a fixed i and k, you have a small matrix over j and l in the total density matrix, and what we have written here is the trace of this small matrix.

We have that:

$$Tr(\rho_1 A_1) = \sum_k \sum_l (\rho_1)_{kl} A_{lk} = \sum_k \sum_j a_{kj}(a_{kj})^*\alpha_k = \langle A_1\rangle \tag{14}$$

So all the expectation values of an operator A_1 (acting only on H_1) are given by:

$$\langle A_1\rangle = Tr\,(\rho_1 A_1) \tag{15}$$

It is not difficult (though somewhat cumbersome) to show that these expressions are independent of basis. The ρ_1 matrix is of course unchanged when we change basis in H_2 (without changing basis in H_1), because the (partial) trace operator does not depend upon the choice of basis. But the ρ_1 operator is also a true operator, in that its expression as a function of basis components transforms correctly under a change of basis in H_1. This means that we can write:

$$\rho_1 = Tr_{H_2}(\rho) \tag{16}$$

and this is a completely basis-independent expression.

3 No signalling in EPR situations.

Let us now consider a typical EPR situation, where we have an entangled state $|\psi\rangle$ as given in the preceding section, and Alice has access to system 1, while Bob has access to system 2. Let us have Bob try to send a message to Alice. Now, in order to do so, Bob can make a choice between applying measurement A_2 or measurement B_2 to system 2, while Alice will perform measurement A_1. If Alice can observe any difference in the expectation value from A_1 according to whether Bob choose to perform A_2 or B_2, then Alice will have received information from Bob through the entangled pair. If not, well, that means that there is no way for Bob to signal anything to Alice, because ANY communication channel can be modelled this way.

Because Alice will measure an expectation value, we will try to find out what happens to ρ_1 when Bob does his thing, because Alice's outcome will be: $\langle A_1\rangle = Tr(\rho_1 A_1)$.

Now, first let us consider that Bob does nothing. We have that

$$\rho_1 = Tr_{H_2}(\rho) \tag{17}$$

with ρ the density matrix of the pure state $|\psi\rangle$. Now, let us assume that Bob performs measurement A_2. We take it that A_2 has as eigenbasis $|f_j\rangle$. The measurement has changed the pure state into the following mixture: we have:

$$|\psi_j\rangle = \sum_i a_{ij}|e_i\rangle|f_j\rangle \tag{18}$$

with weight:

$$\sum_i |a_{ij}|^2 \tag{19}$$

Only, we don't have to introduce explicitly this weight, as it is identical to the normalization factor of $|\psi_j\rangle$ that we didn't introduce. This means that ρ becomes:

$$\rho_a = \sum_j |\psi_j\rangle\langle\psi_j| = a_{kj}|e_k\rangle|f_j\rangle a^*_{lj}\langle e_l|\langle f_j| \tag{20}$$

We see that this density matrix after the measurement of Bob has a special form: only terms in $|f_j\rangle\langle f_j|$ are present (the off-diagonal terms in H_2 have been eliminated, as should be). So we find now as matrix elements:

$$(\rho_a)_{ij,kl} = a_{ij}(a_{kl})^* \delta_{jl} \tag{21}$$

These are the same matrix elements as ρ before the measurement, except that all the elements with $j \neq l$ have been set to 0.

If we now take the partial trace in ρ, in each H_2 submatrix, we use only the elements on the diagonal:

$$\rho_{1,a} = \sum_j a_{ij}(a_{kj})^* \tag{22}$$

But these elements have not been changed! So we see that $\rho_{1,a} = \rho_1$. The reduced density matrix ρ_1 is not influenced whether Bob performed, or didn't perform, a measurement A_2. The reader can object that we had A_2 having a special condition, namely that its eigenvectors where the chosen basis $|f_j\rangle$ of H_2. What happens when Bob performs measurement B_2 ? It should be clear that this doesn't change anything. We are free to change basis in H_2, this doesn't affect the values of ρ_1. One can try to work this out in detail, but the mathematical reason is that each submatrix for a given i and k is a matrix in H_2, and that a change of basis in H_2 will transform this submatrix. The trace, however, is invariant under a change of basis, so the same value will be found for the same i and k. In other words, ρ_1 will consist of exactly the same elements, whether we work it out in the basis $|f_j\rangle$ of H_2, or another basis $|g_j\rangle$.

As a conclusion, we can say that Alice will not be able to determine whether or not Bob performs a measurement, and what measurement he performs, concerning ALL expectation values she can measure locally.