Homework #1

(5pts each unless otherwise noted)

1.1 #7 Write down a differential equation of the form $\frac{dy}{dt} = ay+b$ whose solutions approach $y = 3$ as $t \to \infty$.

Solution. Look carefully at the equation $\frac{dy}{dt} = ay+b$. Take *any* interval of time $[t_0, t_1]$ and let $y_0 = y(t_0)$ and $y_1 = y(t_1)$. If y had gotten bigger between t_0 and t_1, then $y_1 > y_0$. Then using our differential equation ($\frac{dy}{dt} = ay + b$), we see that $y_1 > y_0$ implies that $\frac{dy}{dt}$ at t_1 is bigger than $\frac{dy}{dt}$ at t_0. This shows that if a solution y starts out increasing, then it stays increasing for all time. A similar argument shows that if a solution y starts out decreasing, then it stays decreasing for all time. Therefore *every* solution y of a differential equation of the form $\frac{dy}{dt} = ay+b$ is either always increasing or always decreasing.

Now let's apply this observation to our problem. If all solutions of $\frac{dy}{dt} = ay + b$ approach y=3, yet all solutions are either always increasing or always decreasing (by the previous argument) then we can conclude that:

If $y < 3 \Rightarrow$ y is always increasing (towards 3) $\Rightarrow \frac{dy}{dt} > 0 \Rightarrow ay + b > 0$.

If $y > 3 \Rightarrow$ y is always decreasing (towards 3) $\Rightarrow \frac{dy}{dt} < 0 \Rightarrow ay + b < 0$.

Combining these two statements, we find that:

(i) If y=3 $\Rightarrow$ ay+b=0 $\Rightarrow 3 = \frac{-b}{a}$.

(ii) At y=3, the function ay+b changes from positive to negative
$\Rightarrow$ ay+b has a negative slope $\Rightarrow$ a is negative.

Any a and b that satisfy (i) $3 = \frac{-b}{a}$ and (ii) $a < 0$ will solve this problem:

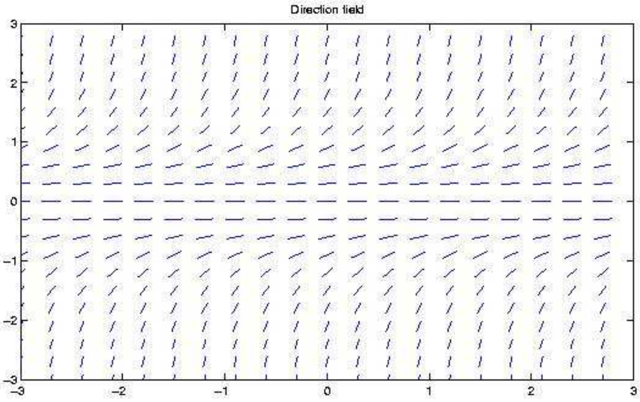
$\frac{dy}{dt} = -y + 3$ is one possible answer [here a=-1 and b=3],

$\frac{dy}{dt} = -2y + 6$ is another [here a=-2 and b=6],

$\frac{dy}{dt} = -\frac{1}{3}y + 1$ is another [here $a = -\frac{1}{3}$ and b=1], etc.

1.1 #13 Draw a direction field for the differential equation $y\prime = y^2$. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

Solution. ***To draw the direction field on a TI-86 or TI-89,*** go to MODE and put the Graph feature to "Diff Eq". Then go to Y= and enter the diff eq, then hit GRAPH. (For a TI-86, you enter the diff eq using Q1 for y. For a TI-89, you enter the diff eq using y1 for y.)



You can see by examining this direction field, that the behavior of any solution as $t \to \infty$, does depend on the initial value. If $y(0) > 0$ then $y \to \infty$ as $t \to \infty$. If $y(0) \leq 0$ then $y \to 0$ as $t \to \infty$.

1.1 #22 A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Solution. When the spherical raindrop evaporates, its *volume* V decreases, so the rate that it evaporates is $\frac{dV}{dt}$. The first sentence of this problem *is* a differential equation!

$$\frac{dV}{dt} = -cA, \tag{1}$$

where c is the constant of proportionality, A is the surface area of the raindrop, and the negative sign is there to remind us that the change is a decrease. We are done, except that the question asks for the differential equation to be a function of time (not of A). Also recall that a differential equation is, *by definition*, an equation involving t, V and V's derivatives. So we rewrite equation (1):

$$\begin{aligned}\frac{dV}{dt} &= -c(4\pi r^2)\\ &= -3c(\frac{4}{3}\pi(r^3)^{\frac{2}{3}})\\ &= -3c(\frac{4}{3}\pi)^{\frac{3}{2}}(\frac{4}{3}\pi r^3)^{\frac{2}{3}}\\ &= -3c(\frac{4}{3}\pi)^{\frac{3}{2}}V^{\frac{2}{3}}\end{aligned}$$

Therefore the differential equation for the volume of the raindrop is $\frac{dV}{dt} = -kV^{\frac{3}{2}}$, where we consolidate all of the constants into one constant, k.

1.1 #23 Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). Suppose that the ambient temperature is 70°F and that the rate constant is $0.05(min)^{-1}$. Write a differential equation for the temperature of the object at any time.

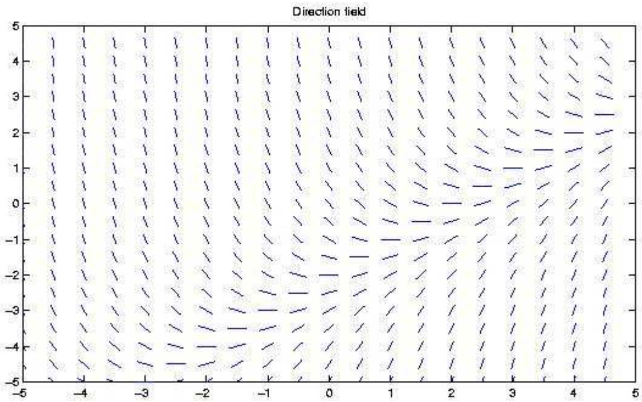
Solution Again, the first sentence of this problem *is* a differential equation:

$$\frac{dT}{dt} = -c(T - T_a), \tag{2}$$

where T is the temperature of the object, c is the constant of proportionality, T_a is the ambient air temperature, and the negative sign denotes cooling (temperature decreasing). The problem goes on to say that $T_a = 70°F$ and c=.05/min, therefore the differential equation for the temperature of the object at any time is:

$$\frac{dT}{dt} = -.05(T - 70).$$

1.1 #26 Draw a direction field for the differential equation $y\prime = -2 + t - y$. Based on the direction field determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.



Solution. You can see by examining this direction field, that the behavior of any solution as $t \to \infty$, does not depend on the initial value. All solutions approach the line y=t-3 (which is also a solution) as $t \to \infty$.

1.2 #4 Consider the differential equation $\frac{dy}{dt} = ay - b$.
(a)Find the equilibrium solution y_e.
(b)Let $Y(t) = y - y_e$; thus Y(t) is the deviation from the equilibrium solution. Find the differential equation satisfied by Y(t).

Solution (a). An equilibrium solution is one in which there is no change, so is where $\frac{dy}{dt} = 0$, i.e. $y_e = \frac{b}{a}$.

Solution (b). To find the differential equation satisfied by Y(t), differentiate Y(t) and use the formula for $\frac{dy}{dt}$:

$$\begin{aligned}\frac{dY}{dt} &= \frac{d}{dt}(y - y_e), \text{ by the definition of Y(t)} \\ &= \frac{d}{dt}(y - \frac{b}{a}), \text{ since } y_e = \tfrac{b}{a} \text{ by part (a)} \\ &= \frac{dy}{dt}, \text{ since } \tfrac{b}{a} \text{ is a constant} \\ &= ay - b, \text{ by the given formula for } \tfrac{dy}{dt} \\ &= a(Y(t) + y_e) - b, \text{ by the definition of Y(t)} \\ &= aY(t), \text{ since } y_e = \tfrac{b}{a}.\end{aligned}$$

Therefore the differential equation satisfied by Y(t) is $\frac{dY}{dt} = aY(t)$.

1.2 #12 A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If Q(t) is the amount present at time t, then $\frac{dQ}{dt} = -rQ$, where r>0 is the decay rate.
(a) If 100mg of thorium-234 decays to 82.04mg in 1 week, determine the decay rate r.
(b) Find an expression for the amount of thorium-234 present at any time t.
(c) Find the time required for the thorium-234 no decay to one-half its original amount.

Solution (a). The first step is to solve the differential equation $\frac{dQ}{dt} = -rQ$. This is an ***equation of the general form*** $\frac{dy}{dt} = ay - b$***. The method for solving it is shown on pg. 12 of the book.*** Follow the exact steps on pg. 12, with a=-r and b=0, and this is what you will get:

$$\frac{\frac{dQ}{dt}}{Q} = -r$$

Integrating both sides,

$$\ln|Q| = -rt + C, \text{ for C constant of integration}$$

Solving for Q,

$$Q = ce^{-rt}$$

Now using the initial condition $Q(0) = 100mg$, solve for c, and get c=100; therefore $Q(t) = 100e^{-rt}$. Now suppose that we measure time in days. After 7 days, the amount of thorium-234 is 82.04, therefore

$$Q(7) = 82.04$$

$$100e^{-r7} = 82.04$$

$$r = \frac{\ln\left(\frac{82.04}{100}\right)}{-7}$$

$$r = 0.02828/day.$$

Solution (b). We already found an equation for the amount of thorium-234 present at any time, in part (a), and we also specified r=0.02828, so $Q(t) = 100e^{-0.02828t}$.

Solution (c). Here we need to find the time t where the original amount of has decayed by half, so when the amount is 50mg. Recall that Q(t) is a function that inputs time and outputs the amount of thorium-234 at that time. Therefore we must solve for t in:

$$Q(t) = 50$$
$$100e^{-0.02828t} = 50$$
$$t = \frac{\ln 50100}{-0.02828}$$
$$t = 24.5days.$$

1.2 #15 According to Newton's law of cooling (see Problem 23 of 1.1), the temperature $u(t)$ of an object satisfies the differential equation $\frac{du}{dt} = -k(u-T)$, where T is the constant ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.
(a)Find the temperature of the object at any time.
(b)Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k and τ.

Solution (a). u(t) is the temperature of the object at any time, and we will find this function by solving the differential equation $\frac{du}{dt} = -k(u-T)$. Rewrite as $\frac{du}{dt} = -ku + kT$, and notice that since k and T are constants, this is of the general form $\frac{dy}{dt} = ay - b$. The method for solving it is shown on pg. 12 of the book. Follow the exact steps on pg. 12, with a=-k and b=-kT, and this is what you will get:

$$\frac{\frac{du}{dt}}{u-T} = -k$$

Integrating both sides,

$$\ln|u-T| = -kt + C, \text{ for C constant of integration}$$

Solving for u,

$$u = ce^{-kt} + T$$

Now using the initial condition $u(0) = u_0$, solve for c, and get $c = u_0 - T$; therefore $u(t) = (u_0 - T)e^{-kt} + T$.

Solution (b). The first sentence of part (b) contains an equation:
temperature difference at time τ = .5(initial temperature difference)
Or, equivalently,

$$u(\tau) - T = .5(u_0 - T)$$
$$(u_0 - T)e^{-k\tau} + T - T = .5(u_0 - T)$$
$$e^{-k\tau} = .5$$
$$k\tau = \ln 2$$

1.3 #1 Determine the order of the differential equation and state whether linear:

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+2y=\sin t.$$

Solution. ***Order*** refers to the highest derivative that appears in the equation (see pg.20 of the book). ***Linearity*** means that the differential equation is of the form

$$a_0(t)\frac{d^ny}{dt^n}+a_1(t)\frac{d^{n-1}y}{dt^{n-1}}+\ldots+a_{n-1}(t)\frac{dy}{dt}+a_n(t)y=g(t)$$

where $a_0(t),\ldots,a_n(t),g(t)$ are functions of t (*not* involving y or y's derivatives)(see pg.21 of the book).
This equation is order 2, linear

1.3 #3 Determine the order of the differential equation and state whether linear:

$$\frac{d^4y}{dt^4}+\frac{d^3y}{dt^3}+\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=1.$$

Solution. Order 4, linear.

1.3 #5 Determine the order of the differential equation and state whether linear:

$$\frac{d^2y}{dt^2}+sin(t+y)=sint.$$

Solution. Order 2, nonlinear.

1.3 #8 Verify that $y_1(t)=e^{-3t}$ and $y_2(t)=e^t$ are solutions to the differential equation $y\prime\prime+2y\prime-3y=0$.

Solution. Checking if $y_1(t)$ is a solution:

$$y_1(t)=e^{-3t}$$

$$\begin{aligned}y_1\prime(t)&=-3e^{-3t}\\ y_1\prime\prime(t)&=9e^{-3t}\\ y_1\prime\prime+2y_1\prime-3y_1&=9e^{-3t}+2(-3e^{-3t})-3e^{-3t}\\ &=(0)(e^{-3t})\\ &=0\end{aligned}$$

Therefore $y_1(t)$ is a solution to the differential equation .
Checking if $y_2(t)$ is a solution:

$$y_2(t)=e^t$$

$$y_2\prime(t)=e^t$$

$$\begin{aligned}
y_2''(t) &= e^t \\
y_2'' + 2y_2' - 3y_2 &= 0 \\
&= e^t + 2e^t - 3e^t \\
&= (0)(e^t) \\
&= 0
\end{aligned}$$

Therefore $y_2(t)$ is also a solution to the differential equation .

1.3 #9 Verify that $y = 3t+t^2$ is a solution to the differential equation $ty'-y = t^2$.

Solution.

$$y(t) = 3t + t^2$$

$$y'(t) = 3 + 2t$$

$$y''(t) = 2$$

$$\begin{aligned}
ty' - y &= t(3 + 2t) - (3t + t^2) \\
&= t^2
\end{aligned}$$

Therefore $y(t)$ is a solution to the differential equation .

1.3 #17 Determine the values of r for which the differential equation $y'' + y' - 6y = 0$ has solutions of the form $y = e^{rt}$.

Solution. If $y = e^{rt}$ is a solution, then:

$$y'' + y' - 6y = 0$$

$$r^2e^{rt} + re^{rt} - 6e^{rt} = 0$$

$$r^2 + r - 6 = 0$$

$$(r + 3)(r - 2) = 0$$

$$r = -3 or 2$$

2.2 #4 Solve the differential equation $y' = (3x^2 - 1)/(3 + 2y)$.

Solution. This is a ***separable equation***. To solve it, collect x's on one side of the equation and y's on the other, then integrate both sides:

$$\frac{dy}{dx} = (3x^2 - 1)/(3 + 2y)$$

$$(3 + 2y)dy = (3x^2 - 1)dx$$

$$\int (3 + 2y)dy = \int (3x^2 - 1)dx$$

$$3y + y^2 = x^3 - x + C,$$

where C is the constant of integration. Now rearrange the last line so that it is in the correct form to use the quadratic equation.

$$y^2 + (3)y + (-x^3 + x - C) = 0$$

Recall the ***quadratic equation*** $y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Using the quadratic equation with $a = 1$, $b = 3$ and $c = (-x^3 + x - C))$,

$$y = \frac{-3 \pm \sqrt{3^2 - 4(-x^3 + x - C)}}{2}$$

$$y = \frac{-3}{2} \pm \sqrt{x^3 - x + k}, y \neq -\frac{3}{2}$$

where all of the constants have been consolidated into one constant k, and where we notice that the differential equation implies that $y \neq -\frac{3}{2}$.

2.2 #6 Solve the differential equation $xy\prime = (1 - y^2)^{1/2}$.

Solution. This is a separable equation. To solve it, collect x's on one side of the equation and y's on the other, then integrate both sides:

$$x\frac{dy}{dx} = (1 - y^2)^{1/2}$$

$$(1 - y^2)^{-1/2}dy = dx/x$$

$$\int (1 - y^2)^{-1/2}dy = \int dx/x$$

$$\arcsin(y) = ln|x| + C \text{for } x \neq 0 \text{ and } |y| \leq 1$$

where C is the constant of integration, and where the restrictions on x and y are because Arcsin and ln are only defined on certain intervals. Now take sine of both sides.

$$y = \sin(ln|x| + C), \text{for } x \neq 0 \text{ and } |y| \ \leq 1$$

2.2 #11 (a)Find the solution of the initial value problem in explicit form:

$$xdx + ye^{-x}dy = 0, y(0) = 1$$

(b)Plot the graph of the solution.
(c)Determine (at least approximately) the interval in which the solution is defined.

Solution (a). Again, this is a separable equation, so collecting x's and y's:

$$-ydy = xe^x dx$$

Integrating both sides,

$$-\frac{1}{2}y^2 = e^x(x - 1) + C$$

To integrate the right-hand side, integration by parts was used.
Recall the ***integration by parts formula***: $\int udv = uv - \int vdu$. (To integrate xe^x, take $u = x$ and $dv = e^x$)
Now that we have integrated, solve for y and get

$$y = \sqrt{2e^x(1-x)+c},$$

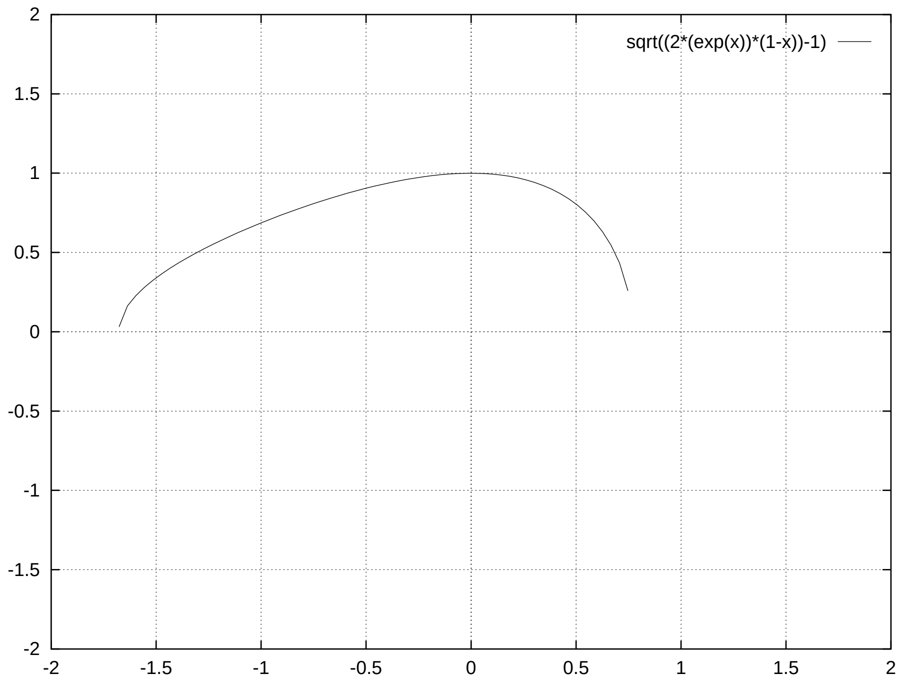
where we have consolidated the constants into a new constant c. Finally, use the initial condition y(0)=1 to find c. So substituting x=0 and y=1, then solving for c:

$$1 = \sqrt{2e^0(1-0)+c}$$

$$c = -1$$

Therefore the explicit solution to the IVP is $y = \sqrt{2e^x(1-x)-1}$.

Solution (b). Graphing the solution $y = \sqrt{2e^x(1-x)-1}$,



Solution (c). By viewing the graph above, we can see that the solution y is only defined approximately for x in the interval (-1.67,.75).

2.2 #18 (a)Find the solution of the initial value problem in explicit form:

$$y\prime = (e^{-x} - e^x)/(3+4y), y(0) = 1$$

(b)Plot the graph of the solution.
(c)Determine (at least approximately) the interval in which the solution is defined.

Solution (a). Again, this is a separable equation, so collecting x's and y's:

$$(e^{-x} - e^{x})dx = (3 + 4y)dy$$

Integrating both sides,

$$(-e^{-x} - e^{x}) = 3y + 2y^2 + C$$

Rearranging so that we can use the quadratic equation,

$$(2)y^2 + (3)y + (e^{-x} + e^{x} + C) = 0$$

Using the quadratic equation,

$$y = \frac{-3 \pm \sqrt{3^2 - 4(2)(e^{-x} + e^{x} + C)}}{(2)(2)},$$

$$y = \frac{-3}{4} \pm \frac{1}{4}\sqrt{9 - 8(e^{-x} + e^{x} + C)},$$

Plugging in the initial condition y(0)=1,

$$1 = \frac{-3}{4} \pm \frac{1}{4}\sqrt{9 - 8(e^{0} + e^{0} + C)},$$

Solving for C,

$$C = -7$$

Therefore the explicit solution to the IVP is

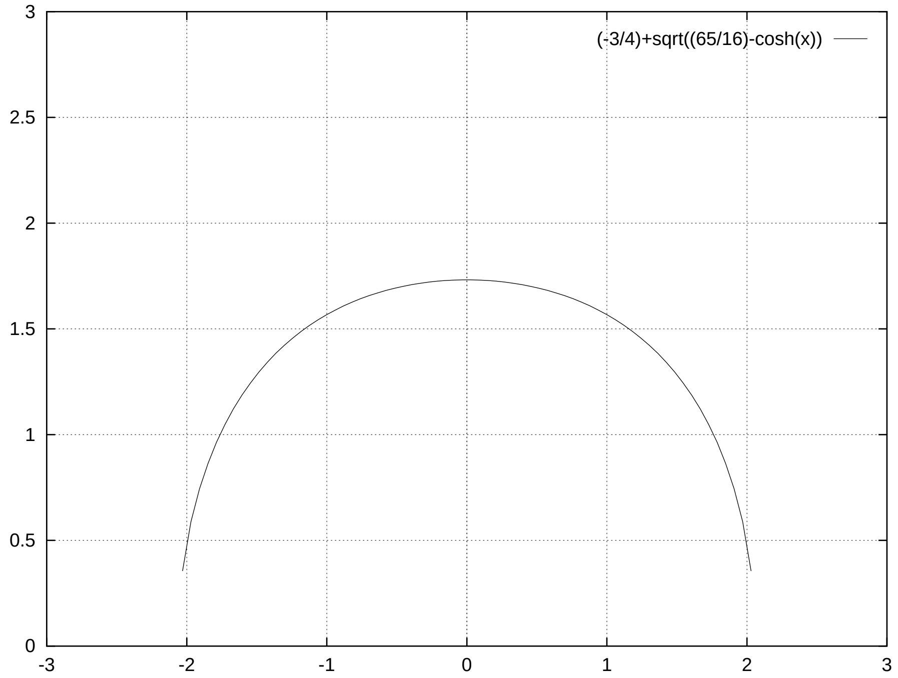
$$y = \frac{-3}{4} + \frac{1}{4}\sqrt{65 - 8(e^{-x} + e^{x})}.$$

Or, equivalently,

$$y = \frac{-3}{4} + \sqrt{65/16 - \cosh x}.$$

(Recall the ***definition of cosh***:=$\frac{e^{x}+e^{-x}}{2}$) (Also notice that in the process of solving for C, we find that the $\pm$ must be a plus)

Solution (b). Graphing the solution $y = \frac{-3}{4} + \sqrt{65/16 - \cosh x}$,



Solution (c). By viewing the graph above, we can see that the solution y is only defined approximately for x in the interval (-2.1,2.1).

2.2 #24 Solve the initial value problem $y\prime = (2 - e^x)/(3 + 2y), y(0) = 0$ and determine where the solution attains its maximum value.

Solution. Again, this is a separable equation, so collecting x's and y's:

$$(3 + 2y)dy = (2 - e^x)dx$$

Integrating both sides,

$$3y + y^2 = 2x - e^x + C$$

Rearranging so that we can use the quadratic equation,

$$y^2 + (3)y + (e^x - 2x - C) = 0$$

Using the quadratic equation,

$$y = \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 - 4(e^x - 2x - C)},$$

Plugging in the initial condition y(0)=0,

$$0 = \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 - 4(e^0 - 0 - C)},$$

Solving for C,

$$C = 1$$

Therefore the explicit solution to the IVP is

$$y = \frac{-3}{2} + \sqrt{-e^x + 2x + \frac{13}{4}}$$

(Notice that in the process of solving for C, we find that the $\pm$ must be a plus)

Graphing the solution $y = \frac{-3}{2} + \sqrt{-e^x + 2x + \frac{13}{4}}$ (see graph below), we see that it attains its maximum value at approximately $x = 1.7$. To find *exactly* where the maximum occurs, notice that the maximum is the only place on the graph where the slope $y\prime$ is zero. So setting $y\prime = 0$,

$$0 = (2 - e^x)/(3 + 2y)$$

$$0 = 2 - e^x$$

Therefore $x = \ln 2$ is the exact place where y attains its maximum.