Othman
HW 10 - April 2019

Problem 1 :
Calculate the transmission and reflection coefficients as functions of the incident angle for right handed circularly polarized light.
Are they the same or different for left handed circularly polarized light?

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Transmission $\equiv \frac{j'}{j} = \frac{energy\ flux\ of\ transited\ wave}{energy\ flux\ of\ the\ incident\ wave}$ ;

Reflection $\equiv \frac{j''}{j} = \frac{energy\ flux\ of\ trans.\ wave}{energy\ flux\ of\ inc.\ wave}$ ;

Energy flux $\equiv j = Re[\overrightarrow{S}] = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\,\vec{n}$ ; where $\vec{n} = \vec{z}$

$$T = \frac{j'}{j} \qquad\qquad R = \frac{j''}{j}$$

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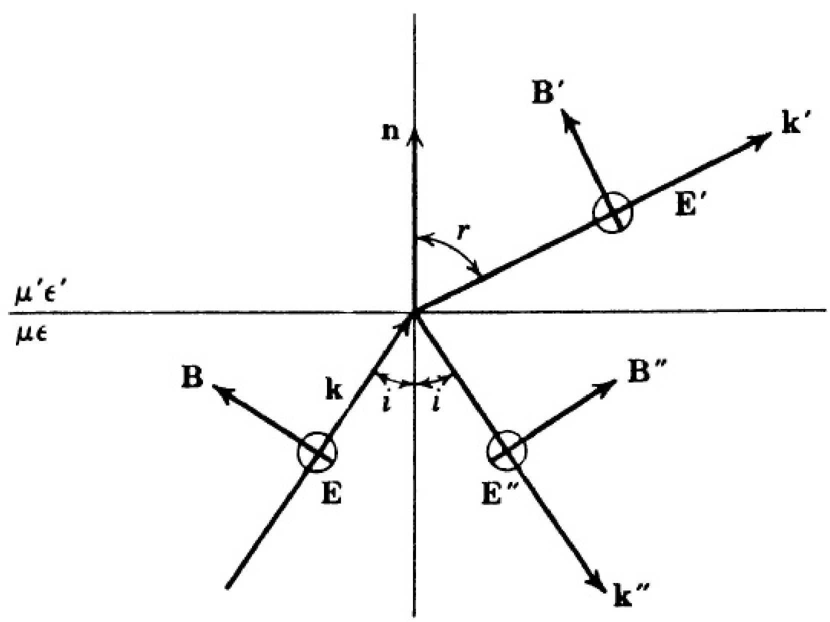


**Figure 1:** E-field is normal to the plane of inc.

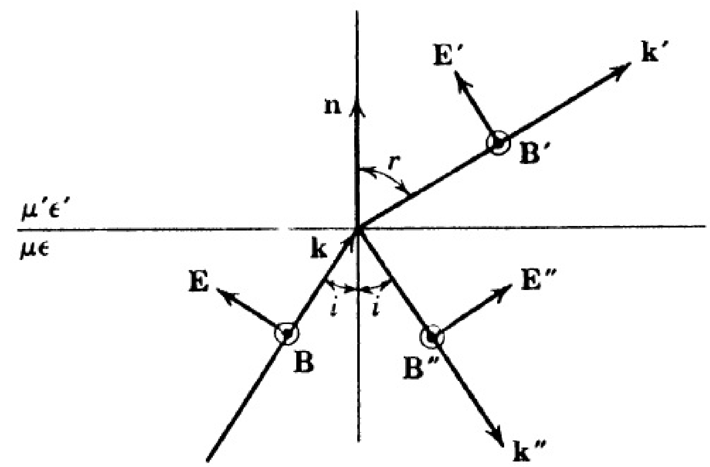


**Figure 2:** E-field is normal to the plane of inc.

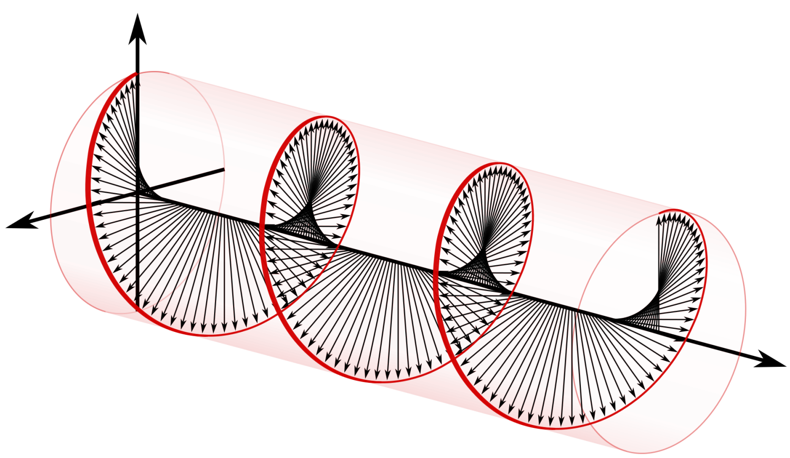

**Figure 3:** Propagation in z-direction

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Right hand circular polarized Electric field :

$$\overrightarrow{E_R} = E_0 \left(\widehat{x} - i\widehat{y}\right) e^{i\left(\vec{k}.\vec{z}-\omega t\right)}$$

Left hand circular polarized Electric field :

$$\overrightarrow{E_L} = E_0 \left(\widehat{x} + i\widehat{y}\right) e^{i\left(\vec{k}.\vec{z}-\omega t\right)}$$

Note : when a circular polarized light enter a median, the transmitted wave and the reflected wave do not have to be circular, in general it is elliptical :

$$\overrightarrow{E_e} = \left(E_x^{'}\,\hat{x} \pm E_y^{'}\,\hat{y}\right) e^{i\left(\vec{k}.\vec{z}-\omega t\right)}$$

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Boundary Conditions (in term of $\overrightarrow{E_0}$) :

$$[1] : D_{\perp}\ continuous$$
$$[2] : B_{\perp}\ continuous$$
$$[3] : E_{\parallel}\ continuous$$
$$[4] : H_{\parallel}\ continuous$$

$$[\epsilon(\mathbf{E}_0 + \mathbf{E}_0'') - \epsilon'\mathbf{E}_0'] \cdot \mathbf{n} = 0$$
$$[\mathbf{k} \times \mathbf{E}_0 + \mathbf{k}'' \times \mathbf{E}_0'' - \mathbf{k}' \times \mathbf{E}_0'] \cdot \mathbf{n} = 0$$
$$(\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0') \times \mathbf{n} = 0$$
$$\left[\frac{1}{\mu}(\mathbf{k} \times \mathbf{E}_0 + \mathbf{k}'' \times \mathbf{E}_0'') - \frac{1}{\mu'}(\mathbf{k}' \times \mathbf{E}_0')\right] \times \mathbf{n} = 0$$

**Figure 4:** in order: [1] , [2] , [3] , and [4]

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Applying the boundary conditions to E-field :

Note : on (z = 0) plane, $\vec{k}\,.\,\vec{z} = 0$ for all waves $\rightarrow e^{i\left(\vec{k}.\vec{z}-\omega t\right)}$ = 1

[1] :

$$\left[\epsilon\left(\overrightarrow{E_i} + \overrightarrow{E_r}''\right) - \epsilon'\,\overrightarrow{E_t}'\right].\,\hat{z} = 0$$

$$\left[\epsilon\left(E_0\left(\hat{x} - i\hat{y}\right) + \left(E_x''\,\hat{x} - E_y''\,\hat{y}\right)\right) - \epsilon'\left(E_x^{'}\,\hat{x} - E_y^{'}\,\hat{y}\right)\right]_{\perp}.\,\hat{z} = 0$$

$$\left[\epsilon\left(E_0\left(i\hat{y}\right)cos(i) + E_y''\left(\hat{y}\right)cos(i)\right) - \epsilon'\left(E_y^{'}\,\hat{y}\right)cos(r)\right] = 0$$

$$\boxed{\epsilon\,(E_0 + E_y'')\,cos(i) - \epsilon'\,E_y^{'}\,cos(r) = 0}$$

[2] :

$$\left[\vec{K} \times \vec{E_i} + \vec{K}'' \times \vec{E_r}'' - \vec{K}' \times \vec{E_t}''\right] . \hat{z} = 0$$

$$\left[k\,\hat{z} \times E_0\left(\hat{x} - i\hat{y}\right) + k''\,\hat{z} \times \left(E_x''\,\hat{x} - E_y''\,\hat{y}\right) - k'\,\hat{z} \times \left(E_x'\,\hat{x} - E_y'\,\hat{y}\right)\right] . \hat{z} = 0$$

$$\left[k\,E_0\left(\hat{z} \times \hat{x} - i\hat{z} \times \hat{y}\right) + k''\left(E_x''\,\hat{z} \times \hat{x} + E_y''\,\hat{z} \times \hat{y}\right) - k'\left(E_x'\,\hat{z} \times \hat{x} - E_y'\,\hat{z} \times \hat{y}\right)\right] . \hat{z} = 0$$

$$\left[k\,E_0\left(\hat{y} + i\hat{x}\right) + k''\left(E_x''\,\hat{y} + E_y''\,\hat{x}\right) - k'\left(E_x'\,\hat{y} - E_y'\,\hat{x}\right)\right] . \hat{z} = 0 \checkmark$$

[3] :

$$\left[\vec{E_i} + \vec{E_r}'' - \vec{E_t}'\right] \times \hat{z} = 0$$

$$\left[E_0\left(\hat{x} - i\hat{y}\right) + \left(E_x''\,\hat{x} + E_y''\,\hat{y}\right) - \left(E_x'\,\hat{x} - E_y'\,\hat{y}\right)\right] \times \hat{z} = 0$$

$$\left[E_0\left(\hat{x} \times \hat{z} - i\hat{y} \times \hat{z}\right) + \left(E_x''\,\hat{x} \times \hat{z} + E_y''\,\hat{y} \times \hat{z}\right) - \left(E_x'\,\hat{x} \times \hat{z} - E_y'\,\hat{y} \times \hat{z}\right)\right] = 0$$

$$\left[E_0\left(-\hat{y} - i\hat{x}\right) + \left(-E_x''\,\hat{y} + E_y''\,\hat{x}\right) - \left(-E_x'\,\hat{y} - E_y'\,\hat{x}\right)\right] = 0$$

$$\left[-E_0\left(\hat{y} + i\hat{x}\right) - \left(E_x''\,\hat{y} - E_y''\,\hat{x}\right) + \left(E_x'\,\hat{y} + E_y'\,\hat{x}\right)\right] = 0$$

$$E_0\left(\hat{y} + i\hat{x}\right) + \left(E_x''\,\hat{y} - E_y''\,\hat{x}\right) = \left(E_x'\,\hat{y} + E_y'\,\hat{x}\right)$$

| | |
|---|---|
| $\perp$ | $E_0\left(\hat{x}\right) - E_y''\left(\hat{x}\right) = E_y'\left(\hat{x}\right)$ |
| $\parallel$ | $E_0\left(\hat{y}\right) + E_x''\left(\hat{y}\right) = E_x'\left(\hat{y}\right)$ |

[4] :

$$\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E_i}+\vec{k}''\times\vec{E_r}\right)-\frac{1}{\mu'}\left(\vec{k}'\times\vec{E_t'}\right)\right]\times\vec{n}=0$$

$$\left[\frac{1}{\mu}\left(k\,E_0\left(\hat{y}+i\hat{x}\right)+k''\left(E_x''\,\hat{y}-E_y''\,\hat{x}\right)\right)-\frac{1}{\mu'}k'\left(E_x'\,\hat{y}+E_y'\,\hat{x}\right)\right]_{\parallel}\times\vec{z}=0$$

$$\left[\frac{1}{\mu}(k\,E_0\,(cos(i)-i)+k''\,(E_x''\,cos(i)+E_y''))-\frac{1}{\mu'}k'\left(E_x'\,cos(r)+E_y'\right)\right]=0$$

| | |
|---|---|
| $\perp$ | $\frac{k}{\mu}E_0\,cos(i)+\frac{k''}{\mu}E_x''\,cos(i)=\frac{k'}{\mu'}E_x'\,cos(r)$ |
| $\parallel$ | $-i\frac{k}{\mu}E_0+\frac{k''}{\mu}E_y''=\frac{k'}{\mu'}E_y'$ |

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Using : Energy flux $\equiv j = Re[\vec{S}] = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\,\vec{n}$ :

$$T=\frac{j'}{j}=\frac{|S'|}{|S|}=\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{E_0'}{E_0}\right)^2 sin^2(i)\Rightarrow$$

| | | |
|---|---|---|
| $\perp$ | $T_\perp$ | $\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{E_x'}{E_0}\right)^2 sin^2(i)$ |
| $\parallel$ | $T_\parallel$ | $\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{E_y'}{E_0}\right)^2 sin^2(i)$ |

$$R=\frac{j''}{j}=\frac{|S''|}{|S|}=\sqrt{\frac{\epsilon''}{\mu}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{E''_0}{E_0}\right)^2 sin^2(i)=\sqrt{\frac{\epsilon''}{\epsilon}}\left(\frac{E''_0}{E_0}\right)^2 sin^2(i)\Rightarrow$$

| | | |
|---|---|---|
| $\perp$ | $R_{\perp}$ | $\sqrt{\frac{\epsilon''}{\epsilon}}\left(\frac{E''_x}{E_0}\right)^2 sin^2(i)$ |
| $\parallel$ | $R_{\parallel}$ | $\sqrt{\frac{\epsilon''}{\epsilon}}\left(\frac{E''_y}{E_0}\right)^2 sin^2(i)$ |

The last step is to find the values of the electric field components from the boundary conditions :

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I have the following equations:

| | |
|---|---|
| | $\epsilon\ (E_0\ +\ E_y'')\ cos(i)\ -\ \epsilon'\ E_y'\ \ cos(r)\ \ =\ 0$ |
| $\perp$ | $E_0\left(\hat{x}\right)\ -\ E_y''\left(\hat{x}\right)\ =\ E_y'\left(\hat{x}\right)$ |
| $\parallel$ | $E_0\left(\hat{y}\right)\ -\ E_x''\left(\hat{y}\right)\ =\ E_x'\left(\hat{y}\right)$ |
| $\perp$ | $\frac{k}{\mu}E_0\ cos(i)\ +\frac{k''}{\mu}\ E_x''\ cos(i)\ =\ \frac{k'}{\mu'}\ E_x'\ cos(r)$ |
| $\parallel$ | $-\ i\frac{k}{\mu}E_0\ \ +\frac{k''}{\mu}\ E_y''\ =\ \frac{k'}{\mu'}\ E_y'$ |

Also, To get The answer in term of the incident angle, I need the following equations:

$$k\ sin(i)\ =\ k'\ sin(r)$$
$$sin(r)\ =\ \frac{k}{k'}\ sin(i)$$
$$k\ =\ \sqrt{\mu\epsilon}\ =\ n$$
$$cos(r)\ =\ \sqrt{1\ -\ sin^2(r)}\ =\ \sqrt{1\ -\ \frac{sin^2(i)}{n^2}}$$

the $\perp$ case :

| | |
|---|---|
| $\perp$ | $E_0 - E_x'' = E_x'$ |
| $\perp$ | $\frac{k}{\mu}E_0 \cos(i) + \frac{k''}{\mu} E_x'' \cos(i) = \frac{k'}{\mu'} E_x' \cos(r)$ |

$$\Longrightarrow E_0 + E_x'' = \frac{\epsilon'}{\epsilon} E_x' \frac{\cos(r)}{\cos(i)} \Longrightarrow \frac{\epsilon'}{\epsilon} \frac{\cos(r)}{\cos(i)} = 1$$

$$\Longrightarrow \frac{k}{\mu} E_0 + \frac{k''}{\mu} E_x'' = \frac{k'}{\mu'} E_x' \frac{\cos(r)}{\cos(i)}$$

$$k E_0 + k'' E_x'' = \frac{k'}{\mu'} \mu E_x' \frac{\cos(r)}{\cos(i)}$$

$$\frac{k}{k'} E_0 + \frac{k''}{k'} E_x'' = \frac{\mu}{\mu'} E_x' \frac{\cos(r)}{\cos(i)}$$

$$\frac{k}{k'} E_0 + \frac{k''}{k'} (- E_x' + E_0) = \frac{\mu}{\mu'} E_x' \frac{\cos(r)}{\cos(i)}$$

$$\frac{k}{k'} E_0 - \frac{k''}{k'} E_x' + \frac{k''}{k'} E_0 = \frac{\mu}{\mu'} E_x' \frac{\cos(r)}{\cos(i)}$$

$$E_0 \left( \frac{k}{k'} + \frac{k''}{k'} \right) = E_x' \left( \frac{\mu}{\mu'} \frac{\cos(r)}{\cos(i)} + \frac{k''}{k'} \right)$$

$$\frac{E_x'}{E_0} = \frac{\left( \frac{k}{k'} + \frac{k''}{k'} \right)}{\left( \frac{\mu}{\mu'} \frac{\cos(r)}{\cos(i)} + \frac{k''}{k'} \right)} \; ; \; k = k''$$

$$\frac{E_x{}'}{E_0} = \frac{2\frac{k}{k'}}{\left(\frac{\mu}{\mu'}\frac{\sqrt{1-\frac{sin^2(i)}{n^2}}}{cos(i)} + \frac{k}{k'}\right)}$$

$$\frac{E_x{}'}{E_0} = \frac{E_0 - E_x{}''}{E_0} = 1 - \frac{E_x{}''}{E_0} = \frac{2\frac{k}{k'}}{\left(\frac{\mu}{\mu'}\frac{\sqrt{1-\frac{sin^2(i)}{n^2}}}{cos(i)} + \frac{k}{k'}\right)} =$$

$$\Longrightarrow \frac{E_x{}''}{E_0} = 1 - \frac{2\frac{k}{k'}}{\left(\frac{\mu}{\mu'}\frac{\sqrt{1-\frac{sin^2(i)}{n^2}}}{cos(i)} + \frac{k}{k'}\right)}$$

the || case :

| | |
|---|---|
| \|\| | $E_0 - E_y{}'' = E_y^{'}$ |
| \|\| | $-i\frac{k}{\mu}E_0 + \frac{k''}{\mu}E_y{}'' = \frac{k'}{\mu'}E_y{}'$ |

$$\Longrightarrow E_0 + E_y{}'' = \frac{\epsilon'}{\epsilon}E_y{}'\frac{cos(r)}{cos(i)} \Longrightarrow \frac{\epsilon'}{\epsilon}\frac{cos(r)}{cos(i)} = 1$$

$$\Longrightarrow i\frac{k}{\mu}E_0 + \frac{k''}{\mu}E_y{}'' = \frac{k'}{\mu'}E_y{}'$$

$$i\,k\,E_0 + k''\,E_y{}'' = \frac{k'}{\mu'}\mu\,E_y{}'$$

$$i\frac{k}{k'}E_0 + \frac{k''}{k'}E_y{}'' = \frac{\mu}{\mu'}E_y{}'$$

$$i\,\frac{k}{k'}\,E_0 + \frac{k''}{k'}\,(-\,E_y' + E_0) = \frac{\mu}{\mu'}\,E_y'$$

$$i\,\frac{k}{k'}\,E_0 - \frac{k''}{k'}E_y' + \frac{k''}{k'}\,E_0 = \frac{\mu}{\mu'}\,E_y'$$

$$E_0\left(i\frac{k}{k'} + \frac{k''}{k'}\right) = E_y'\left(\frac{\mu}{\mu'} + \frac{k''}{k'}\right)$$

$$\frac{E_y'}{E_0} = \frac{\left(i\frac{k}{k'} + \frac{k''}{k'}\right)}{\left(\frac{\mu}{\mu'} + \frac{k''}{k'}\right)}\,;\; k = k''$$

$$\frac{E_y'}{E_0} = \frac{\frac{k}{k'}\,(1+i)}{\left(\frac{\mu}{\mu'} + \frac{k}{k'}\right)}$$

$$\frac{E_y'}{E_0} = \frac{E_0 - E_y''}{E_0} = 1 - \frac{E_y''}{E_0} = \frac{\frac{k}{k'}\,(1+i)}{\left(\frac{\mu}{\mu'} + \frac{k}{k'}\right)} =$$

$$\Longrightarrow \frac{E_y''}{E_0} = 1 - \frac{\frac{k}{k'}\,(1+i)}{\left(\frac{\mu}{\mu'} + \frac{k}{k'}\right)}$$

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Substituting these results back in R and T equations:

T = $\frac{j'}{j} = \frac{|S'|}{|S|} = \sqrt{\frac{\epsilon'}{\mu'}}\,\sqrt{\frac{\mu}{\epsilon}}\left(\frac{E_0'}{E_0}\right)^2 sin^2(i) \Rightarrow$

| | | | |
|---|---|---|---|
| $\perp$ | $T_\perp$ | $\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{E'_x}{E_0}\right)^2 sin^2(i)$ | $\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{2\frac{k}{k'}}{\left(\frac{\mu}{\mu'}\frac{\sqrt{1-\frac{sin^2(i)}{n^2}}}{cos(i)}+\frac{k}{k'}\right)}\right)^2 sin^2(i)$ |
| $\parallel$ | $T_\parallel$ | $\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{E'_y}{E_0}\right)^2 sin^2(i)$ | $\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\frac{k}{k'}(1+i)}{\left(\frac{\mu}{\mu'}+\frac{k}{k'}\right)}\right)^2 sin^2(i)$ |

$$R = \frac{j''}{j} = \frac{|S''|}{|S|} = \sqrt{\frac{\epsilon''}{\mu}}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{E''_0}{E_0}\right)^2 sin^2(i) = \sqrt{\frac{\epsilon''}{\epsilon}}\left(\frac{E''_0}{E_0}\right)^2 sin^2(i) \Rightarrow$$

| | | | |
|---|---|---|---|
| $\perp$ | $R_\perp$ | $\sqrt{\frac{\epsilon''}{\epsilon}}\left(\frac{E''_x}{E_0}\right)^2 sin^2(i)$ | $\sqrt{\frac{\epsilon''}{\epsilon}}\left(1-\frac{2\frac{k}{k'}}{\left(\frac{\mu}{\mu'}\frac{\sqrt{1-\frac{sin^2(i)}{n^2}}}{cos(i)}+\frac{k}{k'}\right)}\right)^2 sin^2(i)$ |
| $\parallel$ | $R_\parallel$ | $\sqrt{\frac{\epsilon''}{\epsilon}}\left(\frac{E''_y}{E_0}\right)^2 sin^2(i)$ | $\sqrt{\frac{\epsilon''}{\epsilon}}\left(1-\frac{\frac{k}{k'}(1+i)}{\left(\frac{\mu}{\mu'}+\frac{k}{k'}\right)}\right)^2 sin^2(i)$ |

? will there going to be a difference if the inc. wave was left hand circularly polarized ?

$\Rightarrow$ Because of the symmetry between the right and the left waves, and the method used above. T and R has to be the same as for the right hand circularly polarized wave.