1 Energy current with one body potential

The Hamiltonian is

$$
\begin{aligned}
H &= -\frac{\hbar^2}{2m}\int dr\psi^\dagger(r)\nabla^2\psi(r) + \int drV(r)\psi^\dagger(r)\psi(r) \\
&= \frac{\hbar^2}{2m}\int dr\nabla\psi^\dagger(r)\nabla\psi(r) + \int drV(r)\psi^\dagger(r)\psi(r) = H_0 + V
\end{aligned}
$$

The Hamiltonian density is

$$
h(r) = \frac{\hbar^2}{2m}\nabla\psi^\dagger(r)\nabla\psi(r) + V(r)\psi^\dagger(r)\psi(r) = h_0(r) + v(r)
$$

I now hope to show that $[h_0, V] + [v, H_0] + [v, V] = 0$

First commutator

$$
[h_0, V] = \frac{\hbar^2}{2m}\int drV(r')\left[\nabla\psi^\dagger(r)\nabla\psi(r), \psi^\dagger(r')\psi(r')\right]
$$

$$
\left[\nabla\psi^\dagger(r)\nabla\psi(r), \psi^\dagger(r')\psi(r')\right] = \nabla\psi^\dagger(r)\nabla\left[\psi(r), \psi^\dagger(r')\psi(r')\right] + \left(\nabla\left[\psi^\dagger(r), \psi^\dagger(r')\psi(r')\right]\right)\nabla\psi(r)
$$

$$
\begin{aligned}
\left[\psi(r), \psi^\dagger(r')\psi(r')\right] &= \left\{\psi^\dagger(r'), \psi(r)\right\}\psi(r') - \psi^\dagger(r')\left\{\psi(r'), \psi(r)\right\} \\
&= \delta(r - r')\psi(r)
\end{aligned}
$$

$$
\begin{aligned}
\left[\psi^\dagger(r), \psi^\dagger(r')\psi(r')\right] &= \left\{\psi^\dagger(r), \psi^\dagger(r')\right\}\psi(r') - \left\{\psi^\dagger(r'), \psi(r')\right\}\psi^\dagger(r) \\
&= -\delta(r - r')\psi^\dagger(r)
\end{aligned}
$$

$$
\left[\nabla\psi^\dagger(r)\nabla\psi(r), \psi^\dagger(r')\psi(r')\right] = \nabla\psi^\dagger(r)\nabla\left(\delta(r - r')\psi(r)\right) - \left(\nabla\left(\delta(r - r')\psi^\dagger(r)\right)\right)\nabla\psi(r)
$$

$$
\begin{aligned}
[h_0, V] &= \frac{\hbar^2}{2m}\int drV(r')\left(\nabla\psi^\dagger(r)\nabla\left(\delta(r - r')\psi(r)\right) - \left(\nabla\left(\delta(r - r')\psi^\dagger(r)\right)\right)\nabla\psi(r)\right) \\
&= -\frac{\hbar^2}{2m}\int drV(r')\left(\left(\nabla^2\psi^\dagger(r)\right)\delta(r - r')\psi(r) - \delta(r - r')\psi^\dagger(r)\nabla^2\psi(r)\right) \\
&= -\frac{\hbar^2}{2m}V(r)\left(\left(\nabla^2\psi^\dagger(r)\right)\psi(r) - \psi^\dagger(r)\nabla^2\psi(r)\right) \\
&= i\hbar V(r)\nabla\cdot j(r)
\end{aligned}
$$

Second commutator

$$
[v, H_0] = V(r)\left[\psi^\dagger(r)\psi(r), H_0\right] = -i\hbar V(r)\nabla\cdot j(r)
$$

Third commutator

$$
[v, V] = \int dr'V(r')V(r)\left[\psi^\dagger(r)\psi(r), \psi^\dagger(r')\psi(r')\right] = 0
$$

So we have $[h_0, V] + [v, H_0] + [v, V] = 0$. Hence with an external one body potential we have $[h, H] = [h_0, H_0]$ and the current is unchanged

$$j_E(r) = -\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\right)^2 \left(\nabla\psi^\dagger(r)\nabla^2\psi(r) - \nabla^2\psi^\dagger(r)\nabla\psi(r)\right)$$

Now if the time derivatives are defined as

$$i\hbar\frac{\partial\psi}{\partial t} = [\psi(r), H_0]$$

$$i\hbar\frac{\partial\psi^\dagger}{\partial t} = \left[\psi^\dagger(r), H_0\right]$$

I find the same result as before

$$j_E(r) = -\frac{\hbar^2}{2m}\left(\nabla\psi^\dagger(r)\frac{\partial\psi}{\partial t} + \frac{\partial\psi^\dagger}{\partial t}\nabla\psi(r)\right)$$

However if the derivatives are defined as

$$i\hbar\frac{\partial\psi}{\partial t} = [\psi(r), H]$$

$$i\hbar\frac{\partial\psi^\dagger}{\partial t} = \left[\psi^\dagger(r), H\right]$$

I find

$$j_E(r) = -\frac{\hbar^2}{2m}\left(\nabla\psi^\dagger(r)\frac{\partial\psi}{\partial t} + \frac{\partial\psi^\dagger}{\partial t}\nabla\psi(r)\right) - V(r)j(r)$$