TUTORIAL CLASS 1, WEEK 2

B2) 2010

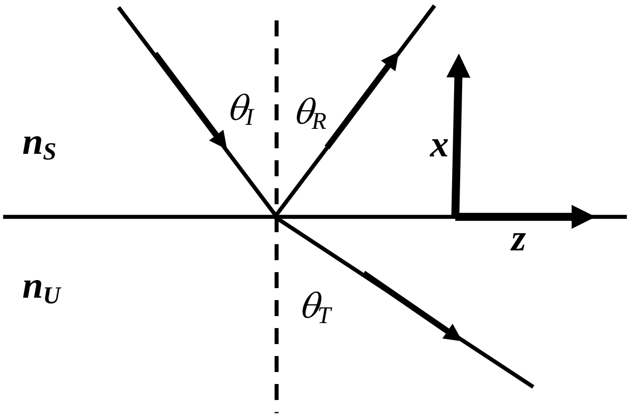


Figure 1

a) Fig. 1 shows light from a sourced region of refractive index n_S incident upon a dielectric interface with an unsourced region of refractive index $n_U < n_S$.

i) What are the boundary conditions on the electric and magnetic fields, *E* and *H*, associated with the light wave?

solution

The boundary conditions for the E and H fields are that the tangential fields (z components) are continuous (equal) across the interface. That is, the sum of the tangential components of the E and H field (superposition) in the sourced region is equal to that in the unsourced region at the interface.

ii) Using these boundary conditions derive an equation for the relationship between the angles of incidence and reflection, θ_I and θ_R. Also derive an equation for the relationship between the angles of incidence and transmission, θ_I and θ_T.

solution

The condition in part i) may be matched at one particular point on the boundary by adjusting the field amplitudes of the incident, reflected and transmitted waves but for the boundary condition to then hold everywhere the individual components of the wavevectors in the plane containing the interface, (the z components), for incident, reflected and transmitted waves, must separately be equal either side of the boundary. For figure 1 this requires $k_{I_z} = k_{R_z} = k_{T_z}$.

For the incident and reflected waves the condition is

$$k_{I_z} = n_S k_0 \sin\theta_I = k_{R_z} = n_S k_0 \sin\theta_R$$

The incident and reflected angles are thus simply related

$$\theta_I = \theta_R$$

For the incident and transmitted waves the condition is

$$k_{I_z} = n_S k_0 \sin\theta_I = k_{T_z} = n_U k_0 \sin\theta_T$$

The incident and refracted angles are thus related

$$n_S \sin\theta_I = n_U \sin\theta_T$$

This is Snell's law.

iii) Beyond the critical angle, $\theta_I > \theta_C$, there will be no transmission of power into the unsourced region but an evanescent electric field will exist, decaying away from the interface into the negative x region (unsourced region). Defining the z component of the wavevector as the propagation coefficient $k_z = \beta$, show that the decay coefficient, α, of this field is given by

$$\alpha = \sqrt{\beta^2 - n_U^2 k_0^2}$$

solution

When $n_S > n_U$ it is possible to have a situation at large enough angle of incidence where

$$k_{I_z} = n_S k_0 \sin\theta_I = \beta > n_U k_0$$

It is now impossible to satisfy the condition $k_{I_z} = n_S k_0 \sin\theta_I = k_{T_z} = n_U k_0 \sin\theta_T$ *if the x component of the wavevector in the unsourced region is real.*
Calling the z component of the wavevector the propagation coefficient β which is the same everywhere we can rewrite the boundary condition as a condition on k_x as follows

$$\beta = \sqrt{k^2 - k_x^2}$$

And the condition is then

$$k_{T_x} = \sqrt{n_U^2 k_0^2 - \beta^2}$$

And therefore

$$k_{T_x} = \pm j\sqrt{\beta^2 - n_U^2 k_0^2} \qquad \theta_I > \theta_C$$

Where $j = \sqrt{-1}$

Writing

$$\alpha = \sqrt{\beta^2 - n_U^2 k_0^2}$$

We have for the x component of the wavevector in the unsourced region

$$k_{T_x} = \pm j\alpha \qquad \theta_I > \theta_C$$

If we write the electric field in the unsourced region as

$$E(z,x) = E_0 \exp(j\beta z)\exp(jk_x x)$$

Where E_0 is the field amplitude then using the expression for $k_{T_x} = \pm j\alpha$ *in this equation;*

$$E(z,x) = E_0 \exp(j\beta z)\exp(\mp \alpha x)$$

We choose the exponentially decaying solution for the unsourced region ($x < 0$) and

$$E(z,x) = E_0 \exp(j\beta z)\exp(\alpha x)$$

$\alpha = \sqrt{\beta^2 - n_U^2 k_0^2}$ *is the decay coefficient.*

b)

i) Write down the **electric field** reflection coefficient for light incident normally at the interface between the sourced and unsourced regions of Fig. 1.

solution

$$r = \frac{\eta_U - \eta_S}{\eta_U + \eta_S} \qquad \text{or} \qquad r = \frac{n_S - n_U}{n_S + n_U}$$

ii) A dielectric medium is placed in air and 20% of the intensity of the light incident normally on the dielectric/air interface is reflected back into the sourced region. If the sourced region is the air calculate the refractive index of the dielectric.

solution
Using the result from i)

$$R = r^2 = \left(\frac{n_S - n_U}{n_S + n_U}\right)^2 = \left(\frac{1 - n_U}{1 + n_U}\right)^2 = 0.2$$

$$\left(\frac{1 - n_U}{1 + n_U}\right) = \sqrt{0.2} = \pm 0.447$$

$$1 - n_U = \pm 0.447(1 + n_U)$$

$$1 \mp 0.447 = (1 \pm 0.447)n_U$$

$$n_U = \frac{1 \mp 0.447}{1 \pm 0.447}$$

A refractive index less than 1 is physically not allowed therefore of the two solutions

$$n_U = \frac{1.447}{0.553} = 2.62$$

iii) Light is incident normally from air onto the surface of water whose refractive index is $n_{H_2O} = 1.334$. Assuming zero absorption of the light what is the percentage of incident power that is transmitted into the water?

solution
The percentage of light reflected is found from the equation of part i)

$$R = r^2 = \left(\frac{n_S - n_U}{n_S + n_U}\right)^2 = \left(\frac{1 - 1.333}{1 + 1.333}\right)^2 = \left(\frac{-0.333}{2.333}\right)^2 = 2.03 \times 10^{-2} = 2\%$$

Therefore as with zero absorption $R + T = 1$

$$T = 98\%$$

c) The amplitude reflection coefficient for TM polarised light incident at an angle θ_I on the interface described in a) is given by Equ 1;

$$r_M^{TM} = \frac{H_R}{H_I} = \frac{\eta_S \cos\theta_I - \eta_U \cos\theta_T}{\eta_S \cos\theta_I + \eta_U \cos\theta_T} \qquad \text{(Equ 1)}$$

i) Using equation 1 and Snell's law show that the Brewster angle, θ_B at which the reflection coefficient for TM polarised waves goes to zero is given by

$$\theta_B = \tan^{-1}\left(\frac{n_U}{n_S}\right)$$

solution

We are given $r_M^{TM} = \frac{H_R}{H_I} = \frac{\eta_S \cos\theta_I - \eta_U \cos\theta_T}{\eta_S \cos\theta_I + \eta_U \cos\theta_T}$ *and need to show that* $r_M^{TM} = 0$ *when* $\theta_I = \theta_B = \tan^{-1}\left(\frac{n_U}{n_S}\right)$.

$$\frac{\eta_S \cos\theta_B - \eta_U \cos\theta_T}{\eta_S \cos\theta_B + \eta_U \cos\theta_T} = 0 \text{ using } \eta = \frac{\eta_0}{n} \qquad \frac{n_U \cos\theta_B - n_S \cos\theta_T}{n_U \cos\theta_B + n_S \cos\theta_T} = 0$$

At the Brewster angle we have

$$\frac{n_U \cos\theta_B - n_S \cos\theta_T}{n_U \cos\theta_B + n_S \cos\theta_T} = 0$$

$$n_U \cos\theta_B = n_S \cos\theta_T$$

Re arranging

$$\cos^2\theta_T = \left(\frac{n_U}{n_S}\right)^2 \cos^2\theta_B$$

Snell's law gives

$$n_S \sin\theta_B = n_U \sin\theta_T$$

$$\sin^2\theta_T = \left(\frac{n_S}{n_U}\right)^2 \sin^2\theta_B$$

Trigonometric identity gives

$$\cos^2\theta_T + \sin^2\theta_T = 1 = \left(\frac{n_S}{n_U}\right)^2 \sin^2\theta_B + \left(\frac{n_U}{n_S}\right)^2 \cos^2\theta_B = \cos^2\theta_B + \sin^2\theta_B$$

Collecting sin and cos terms together

$$\left[\left(\frac{n_S}{n_U}\right)^2 - 1\right]\sin^2\theta_B = \left[1 - \left(\frac{n_U}{n_S}\right)^2\right]\cos^2\theta_B$$

$$\tan^2\theta_B = \frac{1-\left(n_U/n_S\right)^2}{\left(n_S/n_U\right)^2-1} = \frac{\dfrac{n_S^2-n_U^2}{n_S^2}}{\dfrac{n_S^2-n_U^2}{n_U^2}} = \frac{n_U^2}{n_S^2}$$

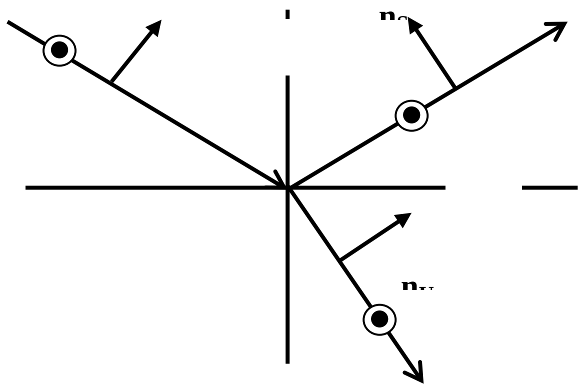
Therefore

$$\theta_B = \tan^{-1}\left(\frac{n_U}{n_S}\right) \qquad \boldsymbol{QED}$$

ii) Using the boundary conditions on both the electric and magnetic fields show that the amplitude reflection coefficient for a TE polarised wave is given by Equ 2

$$r_E^{TE} = \frac{E_R}{E_I} = \frac{\eta_U \sec\theta_T - \eta_S \sec\theta_I}{\eta_U \sec\theta_T + \eta_S \sec\theta_I} \qquad \text{(Equ 2)}$$

solution



At the boundary the BCs require:

$$E_{I_{Tan}} + E_{R_{Tan}} = E_{T_{Tan}}$$

all of the E field is tangential for a TE polarised wave and

$$E_I + E_R = E_T$$

Following the diagram and using the BCs for H

$$H_{I_{Tan}} - H_{R_{Tan}} = H_{T_{Tan}}$$

$$H_I \cos\theta_I - H_R \cos\theta_I = H_T \cos\theta_T$$

Now use the relation between E and H $H = \dfrac{E}{\eta}$ *to obtain the H field boundary conditions in terms of E*

$$\frac{E_I}{\eta_S}\cos\theta_I - \frac{E_R}{\eta_S}\cos\theta_I = \frac{E_T}{\eta_U}\cos\theta_T$$

It is now possible to eliminate E_T from this equation using $E_I + E_R = E_T$

$$\frac{E_I}{\eta_S}\cos\theta_I - \frac{E_R}{\eta_S}\cos\theta_I = \frac{E_I + E_R}{\eta_U}\cos\theta_T$$

$$\frac{(E_I - E_R)}{\eta_S}\cos\theta_I = \frac{(E_I + E_R)}{\eta_U}\cos\theta_T$$

Grouping E_I and E_R on either side this equation

$$E_I(\eta_U\cos\theta_I - \eta_S\cos\theta_T) = E_R(\eta_U\cos\theta_I + \eta_S\cos\theta_T)$$

Finally the electric field amplitude reflection coefficient for the TE polarisation

$$r_E^{TE} = \frac{E_R}{E_I} = \frac{\eta_U\cos\theta_I - \eta_S\cos\theta_T}{\eta_U\cos\theta_I + \eta_S\cos\theta_T}$$

Finally divide top and bottom by $\cos\theta_I \cos\theta_T$ *to obtain*

$$r_E^{TE} = \frac{E_R}{E_I} = \frac{\eta_U\sec\theta_T - \eta_S\sec\theta_I}{\eta_U\sec\theta_T + \eta_S\sec\theta_I} \qquad \boldsymbol{QED}$$

iii) The amplitude transmission coefficient for a TE polarised wave is given by Equ 3

$$t_E^{TE} = \frac{E_T}{E_I} = \frac{2n_S\cos\theta_I}{n_S\cos\theta_I + n_U\cos\theta_T} \qquad \text{(Equ 3)}$$

Using Equ 3 demonstrate that the power transmission coefficient of a TE wave is

$$T = \left(\frac{2n_S\cos\theta_I}{n_S\cos\theta_I + n_U\cos\theta_T}\right)^2 \frac{n_U}{n_S}\frac{\cos\theta_T}{\cos\theta_I}$$

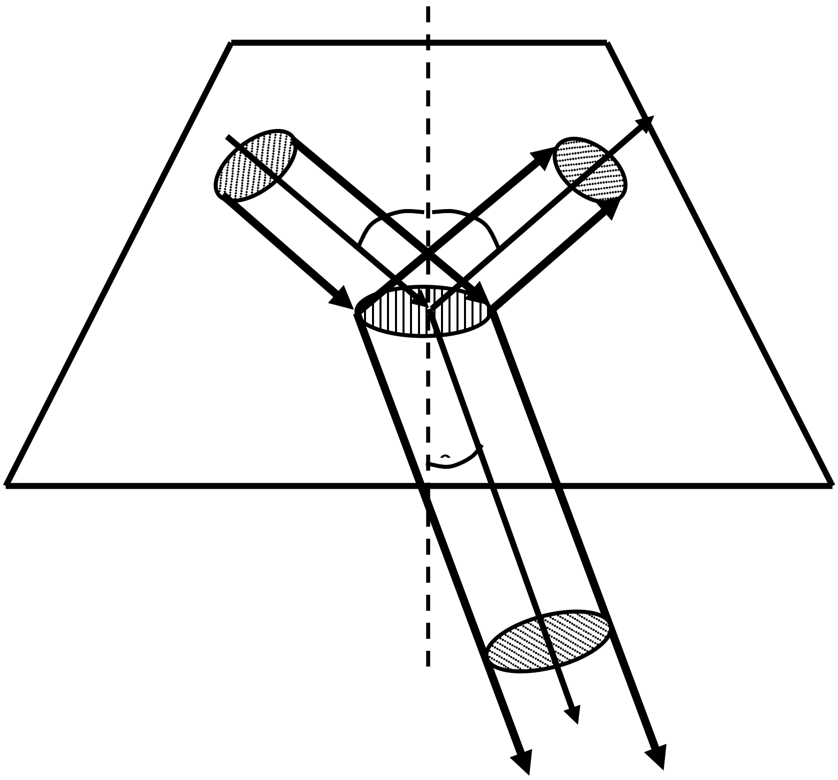
solution

The power is IA where I is the intensity in the wave and A the are normal to the propagation of the wave.

$$P_I = I_I A_I = \frac{E_I^2}{2\eta_S}A\cos\theta_I \qquad P_T = I_T A_I = \frac{E_T^2}{2\eta_U}A\cos\theta_T$$

Where A is the area intercepted by all three waves at the interface.

$$T = \frac{P_T}{P_I} = \frac{E_T^2}{E_I^2}\frac{\eta_S}{\eta_I}\frac{\cos\theta_T}{\cos\theta_I} = t^2\frac{\eta_S}{\eta_I}\frac{\cos\theta_T}{\cos\theta_I}$$



Given that

$$t_E^{TE} = \frac{2n_S \cos\theta_I}{n_S \cos\theta_I + n_U \cos\theta_T}$$

$$T = \left(\frac{2n_S \cos\theta_I}{n_S \cos\theta_I + n_U \cos\theta_T}\right)^2 \frac{\eta_S}{\eta_I} \frac{\cos\theta_T}{\cos\theta_I}$$

$$\eta_S = \frac{\eta_0}{n_S} \qquad\qquad \eta_U = \frac{\eta_0}{n_U}$$

$$T = \left(\frac{2n_S \cos\theta_I}{n_S \cos\theta_I + n_U \cos\theta_T}\right)^2 \frac{n_U}{n_S} \frac{\cos\theta_T}{\cos\theta_I}$$

QED