PHYS 20171 MATHS OF WAVES AND FIELDS: 2007/08 EXAM

Q1(a)

$$g(k) = \frac{1}{2\pi(\alpha + ik)}$$

$$|g(k)|^2 = \frac{1}{4\pi^2(\alpha^2 + k^2)}$$

Your plot should look like a resonance of width 2α, centred on $k = 0$.

Q1(b)

$$N = 4$$

$$a_2 = -2a_0$$

Q1(c)

$$c_1 = 0$$

[Hint: Multiply both sides of the given equation by $P_1(x)$ and integrate.]

Q2(a)

$$b_m = \frac{2}{m\pi}\left(1 - \cos\frac{m\pi}{4}\right)$$

Q2(b)

$$V(r, \phi) = \sum_{m=1}^{\infty} A_m\, r^m \sin(m\phi)$$

[Hint: To exclude $m < 0$, think about the boundary condition on V at the origin.]

Q2(c)

$$V(r, \phi) = \frac{2V_0}{\pi} \sum_{m=1}^{\infty} \frac{1}{m}\left(1 - \cos\frac{m\pi}{4}\right)\left(\frac{r}{a}\right)^m \sin(m\phi)$$

[Hint: By comparing your series for V at $r = a$ with the one in part (a), you shound find that

$$A_m\, a^m = V_0\, b_m, \qquad \text{with the } b_m \text{ given above.]}$$

Q3(a)

$$\omega = c\sqrt{k^2 + \frac{m^2}{a^2}}$$

Q3(b) Eigenfunctions:

$$\Phi(\phi) = \begin{cases} \cos(m\phi), & m = 0, 1, 2, \ldots \\ \sin(m\phi), & m = 1, 2, 3, \ldots \end{cases}$$

Eigenvalues: $m^2 = 0, 1, 4, \ldots$

Q3(c) The wave propagates in the negative z-direction.

$$v_p = c\,\frac{\sqrt{k^2 + m^2/a^2}}{k}$$
$$v_g = c\,\frac{k}{\sqrt{k^2 + m^2/a^2}}$$

For waves with $m = 0$, v_p is constant and hence these waves are *nondispersive*.

Q3(d) For a propagating mode, k^2 must be positive. Using the given frequency in the dispersion relation above, only the modes with $k = 0, 1$ satisfy this condition.

Q4(a) Eigenvalue problem:

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -k^2 X, \qquad \left.\frac{\mathrm{d}X}{\mathrm{d}x}\right|_0 = 0, \qquad X(L) = 0$$

Q4(b)

$$k_n = \frac{n\pi}{2L}, \qquad n = 1, 3, 5, \ldots$$

Q4(c) [Hint: either use the trig identity $\cos A \cos B = \cos(A+B) + \cos(A-B)$, or start from

$$X_n \frac{\mathrm{d}^2 X_m}{\mathrm{d}x^2} - X_m \frac{\mathrm{d}^2 X_n}{\mathrm{d}x^2} = (k_n^2 - k_m^2) X_n X_m,$$

and integrate by parts.]

Q4(d)

$$\gamma_n = D\left(\frac{n\pi}{2L}\right)^2, \qquad n = 1, 3, 5, \ldots$$

At large times, the term with $n = 1$ survives for longest, giving

$$\phi(x,t) \propto \cos\frac{\pi x}{2L}$$