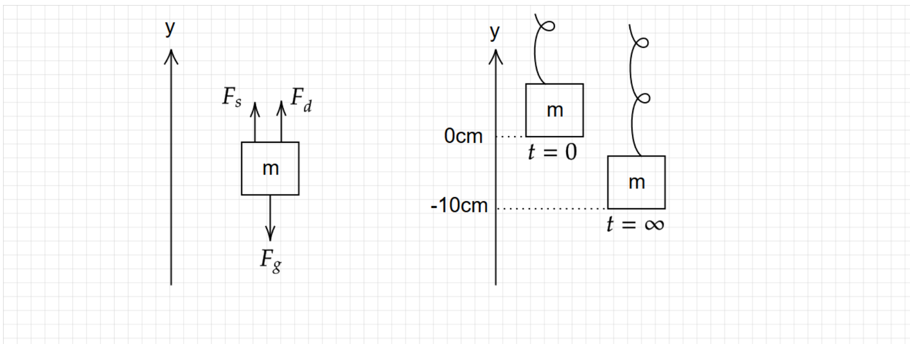
1 Exercise 2.3

A critically mechanical system consisting of a pan hanging from a spring with a damping. What is the value of damping force r if a mass extends the spring by 10cm without overshoot. The mass is 5kg. ($g = 9.81$).



A big thanks to the users of Physics Forum. The solution is taken from this thread: `https://www.physicsforums.com/threads/find-the-resistive-constant-in-a-critically-damped-system.954219/`

1.1 Solution

The three forces that acts on the pan are:

- force of the spring: $F_{\text{spring}} = sy$;
- force of the damping: $F_{\text{damping}} = r\dot{y}$;
- force of gravity: $F_{\text{gravity}} = mg$.

Due to Newton's second law, we have

$$m\ddot{y} = F_{\text{spring}} + F_{\text{damping}} - F_{\text{gravity}} = sy + r\dot{y} - mg$$

We can rewrite this to

$$m\ddot{y} + r\dot{y} + sy = mg$$

The system is critically damped:

$$r^2 - 4ms = 0 \qquad \Rightarrow \qquad s = \frac{r^2}{4m}$$

Then we analyze the situation at $t = \infty$: the pan is in rest position, the velocity is zero so the damping force is zero too. The pan is in equilibrium position so the forces F_{gravity} and F_{spring} are one the opposite of the other. So

$$F_{\text{spring}} = F_{\text{gravity}}$$

$$s\Delta y = mg$$

$$s = \frac{mg}{\Delta y}$$

From the previous formula, we get:

$$r = \sqrt{\frac{s}{4m}} = \sqrt{\frac{4m^2 g}{\Delta y}} = 2m\sqrt{\frac{g}{\Delta y}}$$

Finally we obtain:

$$\ddot{y} + 2\sqrt{\frac{g}{\Delta y}}\dot{y} + \frac{g}{\Delta y}y = g$$

$$\ddot{y} + \quad 2\omega_0 \, \dot{y} \quad + \omega_0^2 \, y = g$$

1.1.1 Solution to the equation of motion

We can't just consider the homogeneous equation because of the constant factor on the right side. We try to guess the solution of the equation:

$$y(t) = e^{-\omega_0 t}(A + Bt) + \frac{g}{\omega_0^2}$$

in order to have

$$\dot{y} = (B - \omega_0)e^{-\omega_0 t}(A + Bt)$$
$$\ddot{y} = (B - \omega_0)^2 e^{-\omega_0 t}(A + Bt)$$

We then substitute into the equation:

$$\left[(B - \omega_0)^2 + 2\omega_0(B - \omega_0) + \omega_0^2\right] e^{-\omega_0 t}(A + Bt) + \omega_0^2 \cdot \frac{g}{\omega_0^2} = g$$
$$\left[B^2 - 2B\omega_0 + \omega_0^2 + 2B\omega_0 - 2\omega_0^2 + \omega_0^2\right] e^{-\omega_0 t}(A + Bt) + g = g$$
$$B^2 \, e^{-\omega_0 t}(A + Bt) = 0$$

We plug in the initial conditions: $y(0) = 0$ because the pan starts from this position in our coordinate system, and $\dot{y}(0) = 0$ because the pan starts in rest.

$$y(0) = 0 \quad \Rightarrow \quad e^0(A + B \cdot 0) + \frac{g}{\omega_0^2} = 0 \quad \Rightarrow \quad A = -\frac{g}{\omega_0^2}$$
$$\dot{y}(0) = 0 \quad \Rightarrow \quad (B - \omega_0)e^0(A + B \cdot 0) = 0 \quad \Rightarrow \quad B = \omega_0$$

The solution of the equation of motion is finally:

$$y(t) = e^{-\omega_0 t} \left(\omega_0 t - \frac{g}{\omega_0^2}\right) + \frac{g}{\omega_0^2}$$