7) If $0<\varepsilon<1$, construct an open set $E\subset[0,1]$ which is dense in $[0,1]$, such that $m(E)=\varepsilon$. (To say that A is dense in B means that the closure of A contains B.) **Proof**: Fix $0<\varepsilon<1$. Construct a sequence closed sets $\{C_k\}$ as follows: Put $C_0=[0,1]$. To form $C_{\lambda+1}$, delete the open middle interval of length $\varepsilon 3^{-\lambda}$ from each complete segment of C_λ; there are $2^{\lambda-1}$ such intervals removed upon each iteration, since removing the middle portion of a segment the leaves two in its place. Define $C=\bigcap_{\lambda=0}^{\infty} C_\lambda$, which, being the intersection of closed sets, is closed. Now define $E=C^C$, where the complement is taken relative to $[0,1]$, so that $E=\left(\bigcap_{\lambda=0}^{\infty} C_\lambda\right)^C=\left(\bigcup_{\lambda=0}^{\infty} C_\lambda^C\right)$ is an open set being formed by the disjoint union of opens sets, whose measures are $m\left(C_0^C\right)=0,$ and $m\left(C_\lambda^C\right)=\frac{\varepsilon}{3}\left(\frac{2}{3}\right)^{\lambda-1}$ for $\lambda\in\mathbb{Z}^+$, and hence

$$m(E)=m\left(\bigcup_{\lambda=0}^{\infty} C_\lambda^C\right)=\sum_{\lambda=0}^{\infty} m\left(C_\lambda^C\right)=\frac{\varepsilon}{3}\sum_{\lambda=0}^{\infty}\left(\frac{2}{3}\right)^{\lambda}=\frac{\frac{\varepsilon}{3}}{1-\frac{2}{3}}=\varepsilon$$

; note that calculation may be made rigorous by applying the limiting process of Theorem 1.19 (d). That E is dense in $[0,1]$ is seen by…