ON MATHEMATICAL INDUCTION*

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Introduction. According to modern standards of logical rigor, each branch of pure mathematics must be founded in one of two ways: either its basic concepts must be *defined* in terms of the concepts of some prior branch of mathematics, in which case its theorems are deduced from those of the prior branch of mathematics with the aid of these definitions, or else its basic concepts are taken as *undefined* and its theorems are deduced from a set of axioms involving these undefined terms.

The natural numbers, 0, 1, 2, 3, $\cdots$ are among those mathematical entities about which we learn at the earliest age, and our knowledge of these numbers and their properties is largely of an intuitive character. Nevertheless, if we wish to establish a precise mathematical theory of these numbers, we cannot rely on unformulated intuition as the basis of the theory but must found the theory in one of the two ways mentioned above. Actually, both ways are possible. Starting with pure logic and the most elementary portions of the theory of sets as prior mathematical sciences, the German mathematician Frege showed how the basic notions of the theory of numbers can be defined in such a way as to permit a full development of this theory. On the other hand the Italian mathematician Peano, taking *natural number*, *zero*, and *successor* as primitive undefined concepts, gave a system of axioms involving these terms which were equally adequate to allow a full development of the theory of natural numbers. In the present paper we shall examine the concept of *definition by mathematical induction* within the framework of Peano's ideas. In this development we shall presuppose only logic and the most elementary portions of the theory of sets; however, we shall find that our subject is greatly illuminated by the introduction of some of the terminology of modern abstract algebra, even though we do not presuppose any of this algebraic material as a basis of our proofs or definitions.

1. Models and the axioms of Peano. It will be convenient here to use the word *model* to refer to a system consisting of a set N, an element 0 of N, and a unary operation† S on N. A model $\langle N, 0, S\rangle$ will be called a *Peano model* if it satisfies the following three conditions (or axioms).

P1. *For all* $x \in N$, $Sx \neq 0$.

P2. *For all* $x, y \in N$, *if* $x \neq y$ *then* $Sx \neq Sy$.

P3. *If G is any subset of N such that* (a) $0 \in G$, *and* (b) *whenever* $x \in G$ *then also* $Sx \in G$, *then* $G = N$.‡

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† A unary operation on N is a function having N as its domain and having a range which is a subset of N. For any $x \in N$, we let Sx be the element of N which is the value obtained by operating on x with S.

‡ In the terminology of set-theory, P1 and P2 respectively express the conditions that 0 is not in the range of S, and that S is one-one. A subset G of N which satisfies condition (b) of P3 is said

If we consider the case when N^* is the set of natural numbers as we know them intuitively, 0^* is the number zero, and S^* is the successor operation (*i.e.*, the operation such that for any natural number x, S^*x is the next following number), then $\langle N^*, 0^*, S^*\rangle$ is an example of a Peano model. Of course it was this example which suggested consideration of the Peano axioms in the first place. Note, however, that there are many other Peano models, for example the system $\langle N', 0', S'\rangle$ where N' is the set of all positive even integers, $0'$ is the number two, and S' is the operation of adding two.

Condition P3 is called the *axiom of mathematical induction*. It will be useful to introduce the term *induction model* to refer to any model which satisfies this axiom. Thus we see at once that every Peano model is an induction model. But the converse is far from true.

For example, let N'' be a set containing a single element, let $0''$ be this element, and let S'' be the only possible unary operation on N'', *i.e.*, the operation such that $S''0''=0''$. Then clearly $\langle N'', 0'', S''\rangle$ is an induction model, but it is not a Peano model since it does not satisfy Axiom P1. For another example let a_0 and a_1 be two distinct objects, and let N''' be the pair $\{a_0, a_1\}$ (*i.e.*, the set having a_0 and a_1 as its only elements). Let S''' be the unary operation on N''' with constant value a_1 (*i.e.*, $S'''x=a_1$ for all $x\in N'''$). Then $\langle N''', a_0, S'''\rangle$ is an induction model, but it is not a Peano model since it does not satisfy Axiom P2.

The reader may notice that the model $\langle N'', 0'', S''\rangle$ satisfies P2, while $\langle N''', a_0, S'''\rangle$ satisfies P1. It then becomes natural to ask whether there are induction models which satisfy *neither* P1 nor P2. As it happens, there are none: any model which satisfies P3 must also satisfy either P1 or P2. A direct proof of this fact, using only the laws of logic and the elements of set theory, is rather troublesome to find—it is a task we leave for the enterprising reader. Presently we shall see that after the theory of definition by mathematical induction is established, the result can be obtained quite simply.

Now consider the following two statements.

P4. *If y is any element of N such that $y\neq Sx$ for all $x\in N$, then $y=0$.*
P5. *For all $x\in N$, $x\neq Sx$.*

We recognize that each of these is true for the Peano model $\langle N^*, 0^*, S^*\rangle$ of natural numbers as we know them intuitively, and in fact we can easily show that each one is true of *all* Peano models by deriving it from Axioms P1–P3. But there is an important difference between the two: the proof of P4 requires

to be *closed* under S. If we regard $\langle N, S\rangle$ as an algebraic system, a subset G of N which is closed under S would be called a *subalgebra* of the system. Thus P3 expresses the condition that the only subalgebra of $\langle N, S\rangle$ which contains the element 0 is N itself. In algebraic terminology this condition is expressed by saying that the element 0 *generates* the algebra $\langle N, S\rangle$. It is also possible to regard the system $\langle N, 0, S\rangle$ itself as an algebraic system. In that case, in order to qualify as a subalgebra a subset G of N must contain 0 as well as be closed under S. Thus P3 expresses the condition that the only subalgebra of $\langle N, 0, S\rangle$ is N itself. Hence the system $\langle N, 0, S\rangle$ is generated by any one of its elements.

only Axiom P3, so that P4 holds for all induction models, but the proof of P5 requires Axiom P1 as well as P3, and our example $\langle N'', 0'', S''\rangle$ shows that P5 does not hold for all induction models.

2. Operations defined by mathematical induction. The statements P4 and P5 are examples of true theorems about natural numbers, but we generally regard their mathematical content as quite trivial. In order to develop a richer theory it is essential to introduce additional concepts beyond the primitive notions of number, zero, and successor; in particular, we must define such central concepts as additition, multiplication, exponentiation, prime number, *etc.* Let us consider addition first, and inquire how it can be defined.

Addition is a binary operation on natural numbers; *i.e.*, it is a function $+$ which may act on any ordered pair $\langle x, y\rangle$ of natural numbers, the result of the action being again a natural number, $x+y$. Peano's idea was to define $+$ by means of the pair of equations

1.1 $$x + 0 = x,$$

1.2 $$x + Sy = S(x + y).$$

Of course on the basis of our intuitive knowledge of the operation of addition we recognize that these equations are true for all natural numbers x and y. But in what sense do the equations constitute a *definition* of addition? In particular, does the definition hold only for natural numbers, or for arbitrary Peano models as well? In order to get a clear answer to these questions we first consider a related but more general problem.

The introduction of an operation by means of the pair of equations 1.1 and 1.2 is an example of what is called *definition by mathematical induction.* To describe this concept in general terms we must consider a Peano model $\langle N, 0, S\rangle$ and in addition a second model $\langle N_1, 0_1, S_1\rangle$ which, however, is not required to be a Peano model (or even an induction model). Being given these two models we say that the pair of equations

2.1 $$h0 = 0_1,$$

2.2 $$h(Sy) = S_1(hy),$$

defines (by mathematical induction) a function h: a function which maps N into N_1 and satisfies 2.1 and 2.2 for all $y \in N$. Again we may raise the question: In what sense do these equations define a function? The answer is provided by the following theorem.

THEOREM I. *No matter what Peano model $\mathfrak{N} = \langle N, 0, S\rangle$ we have, and no matter what model $\mathfrak{N}_1 = \langle N_1, 0_1, S_1\rangle$ we start with, there exists a unique homomorphism of $\mathfrak{N}$ into $\mathfrak{N}_1$; that is, there exists one and only one function h mapping N into N_1 which satisfies* 2.1 *and* 2.2 *for all $y \in N$.*†

† As remarked in footnote ‡, pp. 323–324, a model $\langle N, O, S\rangle$ is an induction model if and only if 0 is a generator of the algebraic system $\langle N, S\rangle$. In algebraic terminology the content of Theorem I

Before attempting to prove this theorem let us see how it applies to the case of 1.1 and 1.2.

Let $\mathfrak{N}$ be an arbitrary Peano model $\langle N, 0, S\rangle$, and for each $x \in N$ let $\mathfrak{N}_x$ be the model $\langle N, x, S\rangle$. Applying Theorem I to the models $\mathfrak{N}$ and $\mathfrak{N}_x$ we see that for each $x \in N$ there will be a unique function h_x mapping N into itself such that the equations

3.1 $$h_x 0 = x,$$

3.2 $$h_x(Sy) = S(h_x y),$$

hold for all $y \in N$. From the existence and uniqueness of these functions h_x we shall show that there exists a unique binary operation of addition on $\langle N, 0, S\rangle$; for later purposes the reader should note that the argument by which we infer the existence and uniqueness of addition from the existence and uniqueness of the functions h_x is purely set-theoretical in character, and does not depend in any way on Axioms P1–P3. Let f be the binary operation on N whose value for any $x, y \in N$ is determined by the equation

4 $$fxy = (h_x y).$$

Using 4, 3.1, and 3.2 we see that f satisfies the equations

5.1 $$fx0 = x,$$

5.2 $$fx(Sy) = S(fxy),$$

for all $x, y \in N$. Furthermore, f is the *only* binary operation on N with this property. For suppose g is any binary operation on N satisfying

6.1 $$gx0 = x,$$

6.2 $$gx(Sy) = S(gxy),$$

for all $x, y \in N$. Then for each $x \in N$ let g_x be the unary operation on N such that

7 $$g_x y = gxy$$

for all $y \in N$. From 7, 6.1, and 6.2 we infer that for any $x \in N$ the equations

8.1 $$g_x 0 = x,$$

8.2 $$g_x(Sy) = S(g_x y),$$

hold for all $y \in N$. Comparing 3.1, 3.2 and 8.1, 8.2 we see that for each $x \in N$ we have $g_x = h_x$, since Theorem I assures us that the function h_x defined by 3.1 and 3.2 is unique. But if $g_x = h_x$ for each $x \in N$, 4 and 7 imply that $fxy = gxy$ for all $x, y \in N$. Since f and g are binary operations on N, we see that $f = g$ by the principle of extensionality.

We have thus inferred from Theorem I that if $\langle N, 0, S\rangle$ is any Peano model,

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would be expressed by saying that in the class of all algebraic systems $\langle N, S\rangle$ the systems $\langle N, S\rangle$ derived from Peano models $\langle N, 0, S\rangle$ are *free*, and in fact are *freely generated by* 0.

there is a unique binary operation f on N satisfying 5.1 and 5.2 for all $x, y \in N$. This binary operation we call addition; if we denote it by the symbol "+", we see that 1.1 and 1.2 are simply another way of writing 5.1 and 5.2. In this way the definition of addition by 1.1 and 1.2 is explained by Theorem I.

We now take up the problem of proving this theorem.

Suppose, then, that $\langle N, 0, S\rangle$ is any Peano model, and that $\langle N_1, 0_1, S_1\rangle$ is an arbitrary model. We wish to show the existence of a unique function h mapping N into N_1 which satisfies 2.1 and 2.2 for all $y \in N$. Putting aside for a moment the question of uniqueness, an argument to show the existence of such a function h is sometimes given along the following lines.

Clearly (the argument goes), h is defined for 0, since $h0 = 0_1$ by 2.1. Furthermore, if h is defined for an element y of N then h is also defined for Sy since $h(Sy) = S_1(hy)$ by 2.2. Thus if we let G be the set of all those $y \in N$ for which h is defined, we see that (a) $0 \in G$, and (b) whenever $y \in G$ then also $Sy \in G$. Applying Axiom P3 we conclude that $G = N$. Thus h is defined for all $y \in N$.

At first sight this argument may seem convincing, but a moment's reflection will suffice to raise doubts. For in this argument we refer to a certain function h. But what is h? Apparently it is a function which satisfies 2.1 and 2.2. Recall, however, that the argument is designed to establish the existence of such a function; clearly, then, it is incorrect to assume in the course of the argument that we have such a function.

This objection may be thought at first to be simply a verbal matter which can be avoided by some minor rewording of the argument. Actually, however, there is something fundamentally wrong with the argument—for the only property of the model $\langle N, 0, S\rangle$ which it employs is Axiom P3! If this argument were essentially correct, it would follow that for any *induction* model $\langle N, 0, S\rangle$, and an arbitrary model $\langle N_1, 0_1, S_1\rangle$, there exists a function h, mapping N into N_1, which satisfies 2.1 and 2.2 for all $y \in N$. But this statement is simply false, as the following example shows.

Let $\mathfrak{N}'' = \langle N'', a_0, S''\rangle$ be the induction model considered in Section 1, in which N'' is the pair of two distinct objects a_0 and a_1, and $S''x = a_1$ for all $x \in N''$. Let T be the unary operation on N'' such that $Ta_0 = a_1$ and $Ta_1 = a_0$, and let $\mathfrak{N}_1$ be the model $\langle N'', a_0, T\rangle$. Now if Theorem I could be applied to the induction model $\mathfrak{N}''$ and the model $\mathfrak{N}_1$, there would exist a mapping h of N'' into itself such that

9.1 $$ha_0 = a_0,$$

9.2 $$h(S''y) = T(hy),$$

for all $y \in N''$. From 9.2 we compute that $h(a_1) = h(S''a_0) = T(ha_0)$, and so by 9.1, $h(a_1) = Ta_0 = a_1$. On the other hand, $h(a_1) = h(S''a_1) = T(ha_1)$ by another application of 9.2, and since we have already computed $ha_1 = a_1$ this shows that $h(a_1) = Ta_1 = a_0$. From $h(a_1) = a_0$ and $h(a_1) = a_1$ we get $a_0 = a_1$, contrary to our hypothesis that a_0 and a_1 are distinct. This contradiction shows that Theorem I does *not* apply to the induction model $\mathfrak{N}''$, and shows *a fortiori* that any proof of this theorem must employ either Axiom P1 or P2, as well as P3; actually, as we shall see later, all three axioms must be employed. We may phrase this observation by saying that the *axiom* of mathematical induction does not itself justify *definitions* by mathematical induction.†

Proof of Theorem I. Let $\mathfrak{N}=\langle N, 0, S\rangle$ be any Peano model and let $\mathfrak{N}_1=\langle N_1, 0_1, S_1\rangle$ be an arbitrary model. A subset H of N is called a *segment* if $0\in H$ and if, whenever $Sx\in H$ then also $x\in H$. By a *partial function* let us mean a function j whose domain is some segment H, which has values in N_1, and which is such that

10.1 $$j0 = 0_1,$$

10.2 $$j(Sx) = S_1(jx)$$

for all x such that $Sx\in H$.

LEMMA 1. *Every element of N is in the domain of some partial function.*

Proof. Let G be the set of those elements of N which are in the domain of some partial function. It is clear that the set $\{0\}$ whose only element is 0 is a segment, since there is no x such that $Sx\in\{0\}$ according to Axiom P1. Furthermore, the function j whose domain is $\{0\}$ and whose value $j0$ is 0_1 is a partial function for the same reason. Hence $0\in G$.

Now suppose that y is any element of G, and let j be a partial function having y in its domain. Let H be the domain of j. If $Sy\in H$ then Sy too is in G; so let us consider the case where $Sy\notin H$. In this case let H' be the subset of N obtained from H by adding Sy as an additional element, and let j' be the function whose domain is H' and whose values are given by the following rule: if $x\in H$ then $j'x=jx$, and if $x=Sy$ then $j'x=S_1(jy)$. We shall show below that j' is a partial function, and hence that $Sy\in G$ in the case that $Sy\notin H$ (as well as in the contrary case considered above). Since G contains 0 and is closed under S, it will follow by Axiom P3 that $G=N$. By definition of G the lemma will therefore be proved.

Thus to complete the proof of the lemma it remains only to show that j' is a partial function. To this end, consider first the domain H' of j'; we have $H'=H\cup\{SY\}$. Since H is the domain of a partial function it is a segment, and hence $0\in H$, so that $0\in H'$. Further, if x is any element of N such that $Sx\in H'$, then also $x\in H'$ as we see by cases: if $Sx\in H$ then $x\in H$ because H is a segment, and if $Sx=Sy$ then $x=y$ by Axiom P2 so that $x\in H$ again (since we know $y\in H$). These considerations show that H' is a segment. Now $j'0=j0=0_1$, and $j'(Sx)=S_1(j'x)$ whenever $Sx\in H'$; the last equation is again established by cases, for if $Sx\in H$ then also $x\in H$ so that $j'(Sx)=j(Sx)$ and $j'x=jx$, while if $Sx=Sy$ then $x=y$ and so $j'(Sy)=S_1(j'y)$. Since j' has a domain which is a seg-

† This fact was clearly brought out by Dedekind in his famous book: Was sind und was sollen die Zahlen? (See "Bemerkung," paragraph 130, section 9.)

ment, and since it satisfies 10.1 and 10.2, it is a partial function. This completes the proof of Lemma 1.

LEMMA 2. *If j_1 and j_2 are partial functions and x is in the domain of each, then $j_1x = j_2x$.*

Proof. Let G be the set of those elements x of N such that $j_1x = j_2x$ whenever j_1 and j_2 are partial functions each of which has x in its domain. Clearly $0 \in G$, since $j0 = 0_1$ for any partial function j. Now suppose x is an arbitrary element of G, and let j_1 and j_2 be any partial functions each of which contains Sx in its domain. Then $j_1(Sx) = S_1(j_1x)$ and $j_2(Sx) = S_1(j_2x)$; but since $x \in G$, $j_1x = j_2x$ and hence $j_1(Sx) = j_2(Sx)$. Therefore $Sx \in G$. Since $0 \in G$ and G is closed under S, it follows by Axiom P3 that $G = N$. Referring to the definition of G we see that Lemma 2 is proved.

By combining Lemmas 1 and 2 we see that for any $x \in N$ there is one and only one $z \in N$ such that $z = jx$ for some partial function j. We let h be the function, with domain N, such that for any $x \in N$ the value hx is this unique $z \in N$. We claim that this function h satisfies 2.1 and 2.2 for all $y \in N$, and that it is the only function with domain N which has this property.

Clearly $h0 = 0_1$ since $j0 = 0_1$ for any partial function j. Furthermore, for any $y \in N$ there is a partial function j such that Sy is in the domain of j (by Lemma 1), whence we see that $h(Sy) = j(Sy) = S_1(jy) = S_1(hy)$, so that h satisfies 2.1 and 2.2 as claimed. Now if h_1 is any other function with domain N which satisfies 2.1 and 2.2 for all $y \in N$, then $h_1 = h$. The reason is that N is clearly a segment, so that h and h_1 are both partial functions, whence by Lemma 2 we have $hx = h_1x$ for all $x \in N$, and therefore $h = h_1$ by the principle of extensionality. This completes the proof of Theorem I.

This proof is rather more involved than the simple argument sketched at first, but it has the advantage of being correct. The reader will notice that all of the Axioms P1–P3 were employed in the proof (in connection with Lemma 1).

The construction of h by means of partial functions, as described in this proof, is not the only means we know of obtaining this function. There is another method of constructing h which leads to a different proof of Theorem I. We will outline this other process of construction briefly, leaving the reader to supply details of the proof.

Being given a Peano model $\langle N, 0, S\rangle$ and an arbitrary model $\langle N_1, 0_1, S_1\rangle$, we consider subsets A of the product set $N \times N_1$, *i.e.*, sets A all of whose elements are ordered pairs $\langle x, y\rangle$ where $x \in N$ and $y \in N_1$. We call such a set A *regular* if $\langle 0, 0_1\rangle \in A$ and if, whenever $\langle x, y\rangle \in A$ then also $\langle Sx, S_1y\rangle \in A$. Clearly there are regular sets, for example $N \times N_1$ itself. It is easy to see that the intersection A^* of all regular sets A is itself regular. Now using Axioms P1–P3, which hold for $\langle N, 0, S\rangle$, we can show that for every $x \in N$ there is one and only one $y \in N_1$ such that $\langle x, y\rangle \in A^*$: this is the part of the proof where the reader will have to supply some detail. Having shown this, we define h to be the function with domain N such that for any $x \in N$, hx is the unique $y \in N_1$ for which $\langle x, y\rangle \in A^*$.

From the fact that A^* is regular it then easily follows that h is a homomorphism of N into N_1, as required by Theorem I. Finally, to show the uniqueness of h, we consider any homomorphism h_1 of N into N_1. Letting B be the subset of $N \times N_1$ such that $\langle x, y\rangle \in B$ if and only if $y = h_1 x$, we easily see that B is regular and hence $A^* \subseteq B$. From the fact that for each $x \in N$ there is only one $y \in N_1$ such that $\langle x, y\rangle \in B$, we then infer that $A^* = B$; and from this we easily conclude that $h_1 = h$. This completes our outline of the second proof of Theorem I.

3. Addition and multiplication in arbitrary induction models. As we have previously noted, we can infer from Theorem I the existence, in every Peano model, of a unique operation of addition (*i.e.*, a binary operation f which satisfies 5.1 and 5.2 for all $x, y \in N$). Actually, something more is true.

THEOREM II. *In every induction model there is a unique operation of addition.*

We cannot hope to prove this theorem by means of Theorem I, for the latter is not true of all induction models, as we have seen in Section 2. Instead, we proceed by means of the following lemma.

LEMMA. *If $\langle N, 0, S\rangle$ is any induction model, then for every $x \in N$ there is a unique unary operation h_x on N such that* 3.1 *and* 3.2 *hold for all $y \in N$.*

Proof. We first observe that for any $x \in N$ there can be at most one operation h_x satisfying 3.1 and 3.2. For suppose h_x and h_x' are both operations which satisfy these equations, and let G be the subset of N such that $y \in G$ if and only if $h_x y = h_x' y$. Clearly $0 \in G$, since $h_x 0 = x = h_x' 0$. Also G is closed under S, for if $y \in G$ (so that $h_x y = h_x' y$), we see that $h_x(Sy) = S(h_x y) = S(h_x' y) = h_x'(Sy)$ whence $Sy \in G$. Since Axiom P3 holds for $\langle N, 0, S\rangle$ we infer that $G = N$. By the principle of extensionality it follows that $h_x = h_x'$.

Now let H be the subset of N consisting of those elements x for which an operation h_x exists. Taking h_0 to be the identity operation on N (such that $h_0 y = y$ for all $y \in N$), we see that $h_0 0 = 0$ (thus satisfying 3.1), and for any $y \in N$, $h_0(Sy) = Sy = S(h_0 y)$ (thus satisfying 3.2). Hence $0 \in H$. Furthermore, H is closed under S. For suppose $x \in H$, so that an operation h_x exists. Let h_{Sx} be the operation on N such that $h_{Sx} y = S(h_x y)$ for all $y \in N$. We see that $h_{Sx} 0 = S(h_x 0) = Sx$ (thus satisfying 3.1 for Sx), while for any $y \in N$, $h_{Sx}(Sy) = S(h_x(Sy)) = S(S(h_x y)) = S(h_{Sx} y)$ (thus satisfying 3.2 for Sx); whence $Sx \in H$. Applying Axiom P3, which holds for $\langle N, 0, x\rangle$ by assumption, we infer that $H = N$, which proves the lemma.

Using this lemma we can complete the proof of Theorem II by exactly the same argument used earlier to infer the existence of an addition operation from Theorem I. The reader will notice that in this earlier argument of Section 2 we used Theorem I to obtain the statement of our present lemma; as noted there, the remainder of the argument was of a general set-theoretical character independent of the axioms P1–P3, and so can be applied to the induction model $\langle N, 0, S\rangle$ of Theorem II.

When we consider multiplication we find a situation entirely analogous to the one we have just encountered for addition. By a multiplication operation for an induction model $\langle N, 0, S\rangle$ we mean a binary operation $\cdot$ on N such that the equations

11.1 $$x\cdot 0 = 0,$$

11.2 $$x\cdot(Sy) = (x\cdot y) + x,$$

hold for all $x, y\in N$, where $+$ is the addition operation for the model. Employing a proof closely patterned after the one given above for Theorem II, we can show that in every induction model there is one and only one multiplication operation. We leave the reader to supply the details of such a proof.

4. Operations in Peano models obtained by primitive recursion. When we come to the operation of exponentiation the situation changes. By an exponential operation for an induction model $\langle N, 0, S\rangle$ is meant a binary operation exp such that the equations

12.1 $$x \exp 0 = S0,$$

12.2 $$x \exp (Sy) = (x \exp y)\cdot x,$$

hold for all $x, y\in N$, where $\cdot$ is the multiplication of the model. This time there is no analogue of Theorem II, for it is simply not true that every induction model has an exponential operation; a counter-example is provided by the model $\langle N'', a_0, T\rangle$ which we have considered in Section 2, where a_0 and a_1 are distinct objects, $N''=\{a_0, a_1\}$, and $Ta_0=a_1$, $Ta_1=a_0$. It is a simple matter to show that if exp were a binary operation on N'' satisfying 12.1 and 12.2 for all $x, y\in N''$, then we would have $a_0 \exp z=a_0$ (for any $z\in N''$) from 12.2 (since $w\cdot a_0=a_0$ for all $w\in N''$), while from 12.1 we get $a_0 \exp a_0=a_1$. This contradiction shows that $\langle N'', a_0, T\rangle$ possesses no exponential operation.

However, by an application of Theorem I we can show that every *Peano* model possesses a unique exponential operation. In fact, we can get a general result of which this is a special case.

THEOREM III. *Let $\langle N, 0, S\rangle$ be any Peano model, let f be a unary operation on N, and let g be a ternary operation on N. Then there exists one and only one binary operation j on N such that the equations*

13.1 $$jx0 = fx,$$

13.2 $$jx(Sy) = gxy(jxy),$$

hold for all $x, y\in N$.†

† This function j is said to be obtained by *primitive recursion* from f and g. More generally, one allows for the possibility of obtaining an n-ary operation j from an $(n-1)$-ary operation f and $(n+1)$-ary operation g, for any $n=1, 2, \cdots$; we have selected the case $n=2$ merely to simplify notation, since the idea of the proof is the same for any n.

(If we take the particular case where f is the unary operation on N such that $fx=S0$ for all $x\in N$, and where g is the ternary operation on N such that $gxyz=z\cdot x$ for all x, y, z, then clearly the function j obtained will be the exponential operation.)

Proof. As remarked in the course of our proof of Theorem II, it is sufficient to show that for each $x\in N$ there is a unique unary operation j_x on N such that

14.1 $$j_x 0 = fx,$$

14.2 $$j_x(Sy) = gxy(j_x y),$$

for all $y\in N$; for then we can infer Theorem III by a general set-theoretical argument which holds for all models (*i.e.*, which does not involve any of the Axioms P1–P3). To this end, let N_1 be the set $N\times N$ of all ordered pairs $\langle y, z\rangle$ for all $y, z\in N$. For each $x\in N$ let 0_x be the element $\langle 0, fx\rangle$ of N_1, and let S_x be the unary operation on N_1 such that $S_x\langle yz\rangle=\langle Sy, gxyz\rangle$ for all $y, z\in N$. Applying Theorem I to the Peano model $\langle N, 0, S\rangle$ and the model $\langle N_1, 0_x, S_x\rangle$, we see that for each $x\in N$ there is a unique mapping h_x of N into N_1 such that the equations

15.1 $$h_x 0 = 0_x,$$

15.2 $$h_x(Sy) = S_x(h_x y),$$

hold for all $y\in N$.

Next let L and R be the mappings of N_1 into N such that $L\langle x, y\rangle=x$ and $R\langle x, y\rangle=y$ for all $x, y\in N$, and for each $x\in N$ let j_x and k_x be the unary operations on N such that $j_x y=R(h_x y)$ and $k_x y=L(h_x y)$ for all $y\in N$. We shall show that j_x satisfies 14.1 and 14.2 for all $y\in N$, but first it is necessary to prove that $k_x y=y$ for all $y\in N$. To do this, let G be the subset of N consisting of those elements y such that $k_x y=y$. Since $k_x 0=L(h_x 0)=L0_x=0$, we see that $0\in G$. Next, let y be any element of G. Then $k_x(Sy)=L(h_x(Sy))=L(S_x(h_x y))$ by 15.2. By definition of S_x we have

$$\begin{aligned} S_x(h_x y) &= \langle S(L(h_x y)),\ gx(L(h_x y))(R(h_x y))\rangle \\ &= \langle S(k_x y),\ gx(k_x y)(j_x y)\rangle, \end{aligned}$$

so that $k_x(Sy)=S(k_x y)$. Since $y\in G$ it follows that $k_x(Sy)=Sy$, whence $Sy\in G$. We have thus shown that G is closed under S, and since Axiom P3 holds for $\langle N, 0, S\rangle$ we conclude that $G=N$, *i.e.*, that $k_x y=y$ for all $y\in N$. Returning to the formula

$$S_x(h_x y) = \langle S(k_x y),\ gx(k_x y)(j_x y)\rangle,$$

derived above, we see that it can now be simplified to

$$S_x(h_x y) = \langle Sy,\ gxy(j_x y)\rangle.$$

Since $j_x(Sy)=R(h_x(Sy))=R(S_x(h_x y))$, we get at once that $j_x(Sy)=gxy(j_x y)$, showing that j_x satisfies 14.2. On the other hand $j_x0 = R(h_x0) = R0_x = fx$, so that j_x also satisfies 14.1. To complete the proof of Theorem III it merely remains to show that j_x is the *only* unary operation on N which satisfies these two equations; this can be done along the same lines of argument which we used when we inferred from Theorem I that there was only one addition operation for any Peano model—as we leave the reader to verify.

5. The relation between Peano models and induction models. Why is it that the operations of addition and multiplication exist in every induction model, while the existence of an exponential operation can be guaranteed only for Peano models? To answer this, we must first understand the relation which holds between Peano models and more general induction models. It turns out that from the algebraic viewpoint this relation is a very simple and natural one.

Let $\mathfrak{N} = \langle N, 0, S\rangle$ and $\mathfrak{N}_1 = \langle N_1, 0_1, S_1\rangle$ be any two models. We say that $\mathfrak{N}_1$ is a *homomorphic image* of $\mathfrak{N}$ if there exists a homomorphism h of $\mathfrak{N}$ *onto* $\mathfrak{N}_1$, *i.e.*, a function h whose domain is N, and whose range is the whole of N_1, such that $h0 = 0_1$ and $h(Sx) = S_1(hx)$ for all $x \in N$.

THEOREM IV. *Let $\mathfrak{N} = \langle N, 0, S\rangle$ be a Peano model and $\mathfrak{N}_1 = \langle N(, 0_1, S_1\rangle$ an arbitrary model. A necessary and sufficient condition that $\mathfrak{N}_1$ be a homomorphic image of $\mathfrak{N}$ is that $\mathfrak{N}_1$ be an induction model.*

Proof of necessity. Suppose that $\mathfrak{N}_1$ is a homomorphic image of $\mathfrak{N}$, and let h be a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_1$. Let G_1 be any subset of N_1 such that $0_1 \in G_1$ and such that G_1 is closed under S_1. To show that $\mathfrak{N}_1$ is an induction model, it suffices to show that $G_1 = N_1$. To this end we consider the subset G of N consisting of just those elements x such that $hx \in G_1$. Since $h0 = 0_1$ we see that $0 \in G$. Furthermore, G is closed under S. For suppose $x \in G$, so that $hx \in G_1$. Now $S_1(hx) \in G_1$ since G_1 is closed under S_1. But $h(Sx) = S_1(hx)$ since h is a homomorphism. Thus $h(Sx) \in G_1$ and so $Sx \in G$; hence G is closed under S. Since $\langle N, 0, S\rangle$ satisfies Axiom P3, we see that $G = N$. This means that $hx \in G_1$ for all $x \in N$. Since h has domain N and range N_1 we infer that $G_1 = N_1$.

Proof of sufficiency. Suppose that $\langle N_1, 0_1, S_1\rangle$ is an induction model. Since $\langle N, 0, S\rangle$ is a Peano model we can apply Theorem I to infer the existence of a (unique) homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}_1$. To complete the proof of our theorem it is only necessary to show that the range of h is the whole of N_1. But clearly the range of h contains 0_1, since $h0 = 0_1$, and it is closed under S_1 since for any element z in the range of h there must be an $x \in N$ for which $hx = z$, whence $S_1z = S_1(hx) = h(Sx)$ so that S_1z is also in the range of h. But $\mathfrak{N}_1$ satisfies Axiom P3, so the range of h is N_1 as was to be shown.

From Theorem IV there follows an important and well-known corollary.

THEOREM V. *Any two Peano models are isomorphic.*†

Proof. Let $\mathfrak{N}$ and $\mathfrak{N}_1$ be any Peano models. By Theorem IV there is a homo-

† That is, there is a homomorphism of one onto the other which is one-one.

morphism h of $\mathfrak{N}$ onto $\mathfrak{N}_1$ and a homomorphism h_1 of $\mathfrak{N}_1$ onto $\mathfrak{N}$. Clearly the composed function (h_1h), which is a unary operation on N such that $(h_1h)x = h_1(hx)$ for all $x \in N$, is a homomorphism of $\mathfrak{N}$ into $\mathfrak{N}$. But by Theorem I there is only one homomorphism of $\mathfrak{N}$ into $\mathfrak{N}$, and obviously the identity operation on $\mathfrak{N}$ is such a homomorphism. Hence (h_1h) is the identity operation: $(h_1h)x = x$ for all $x \in N$. It easily follows that h is one-one, and hence h is an isomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_1$.

The principal significance of Theorem V is a metamathematical consequence to the effect that for a certain large set of sentences containing the symbols "N", "0", and "S", any sentence of the set which is true of one Peano model is also true of every other Peano model; hence if a sentence of this set is true of some particular Peano model such as the system $\langle N^*, 0^*, S^*\rangle$ of natural numbers as we know them intuitively, it will be a logical consequence of the Axioms P1–P3. The set of sentences to which this metamathematical result applies contains all those sentences about models which have any interest for us in the present work. Accordingly, from now on, instead of speaking of an arbitrary Peano model we may speak of the system $\langle N^*, 0^*, S^*\rangle$ of natural numbers.

In Theorem II we have seen that every induction model $\mathfrak{N} = \langle N, 0, S\rangle$ possesses a unique operation of addition, $+$. Now that we see from Theorem IV that $\mathfrak{N}$ bears a close relation to the system of natural numbers $\mathfrak{N}^* = \langle N^*, 0^*, S^*\rangle$, it is natural to inquire into the relation between the operation $+$ of $\mathfrak{N}$ and the addition operation $+^*$ of the system $\mathfrak{N}^*$. The answer to this inquiry is given by the following result.

THEOREM VI. *Let $\mathfrak{N} = \langle N, 0, S\rangle$ be any induction model and $+$ its operation of addition. Let h be the unique homomorphism of $\mathfrak{N}^*$ onto $\mathfrak{N}$. Then $h(x +^* y) = hx + hy$ for all $x, y \in N^*$.*

Proof. Let x be any element of N^*, and let G be the subset of N^* such that $y \in G$ if and only if $h(x +^* y) = hx + hy$. Since $h(x +^* 0^*) = hx = hx + 0 = hx + h0^*$ we see that $0^* \in G$. Now let y be any element of G. Then $h(x +^* y) = hx + hy$, so that $h(x +^* S^*y) = h(S^*(x +^* y)) = S(h(x +^* y)) = S(hx + hy) = hx + S(hy) = hx + h(S^*y)$, and hence $S^*y \in G$. Since G is closed under S^* we apply Axiom P3 (which holds for $\mathfrak{N}^*$) to conclude that $G = N^*$; and this proves the theorem.

In the terminology of set-theory and algebra we express the content of Theorem VI by saying that the operation $+$ on N is the image (under the homomorphism h) of the operation $+^*$ on N^*. The corresponding theorem for multiplication is equally true, and can be proved in essentially the same way. Now there is a well-known result for general algebraic systems to the effect that if an equation is satisfied identically in one system, the same equation will be satisfied in any homomorphic image of the system if each operation of the original system which enters into the equation is replaced by an operation (of the second system) which is its image under the homomorphism. It follows that such identities as the associative, commutative, and distributive laws which hold for $+^*$ and

$\cdot^*$ in the system $\mathfrak{N}^*$ will also hold for the operations $+$ and $\cdot$ of any induction model. Actually this fact can be shown directly, without appeal to any result which holds for general algebraic systems; for upon inspecting the usual derivation of these laws for Peano models, proceeding from 1.1, 1.2, 11.1, and 11.2, it will be found that only Axiom P3, never P1 or P2, is used.

6. Congruence relations. Another fact about homomorphisms which is well known for general algebraic systems, and which can be established directly in a simple manner for the models under investigation here, is their close connection with congruence relations. By a *congruence relation* for a model $\mathfrak{N}=\langle N, 0, S\rangle$ is meant an equivalence relation R on N† such that whenever xRy then also $(Sx)R(Sy)$. If h is any homomorphism of $\mathfrak{N}$ into some other model, then the relation R_h such that for any $x, y\in N$ we have xR_hy if and only if $hx=hy$, is a congruence relation. Conversely, for every congruence relation R of $\mathfrak{N}$ there is a homomorphism h of $\mathfrak{N}$ onto some other model $\mathfrak{N}_R$, such that $R_h=R$. To construct $\mathfrak{N}_R$ we take N_R to be the class of all equivalence sets x_R for all $x\in N$, we let S_R be the operation on N_R (whose existence follows from the fact that R is a congruence relation of $\mathfrak{N}$) such that $S_Rx_R=(Sx)_R$ for all $x\in N$, and we set $\mathfrak{N}_R=\langle N_R, 0_R, S_R\rangle$. If we define h to be the function mapping N onto N_R such that $hx=x_R$ for all $x\in N$, we easily see that h is a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_R$ and that $R_h=R$.

If h_1 and h_2 are homomorphisms of $\mathfrak{N}$ onto models $\mathfrak{N}_1$ and $\mathfrak{N}_2$ respectively, and if $R_{h_1}=R_{h_2}$, then $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are isomorphic. It follows that every homomorphic image of a model $\mathfrak{N}$ is isomorphic to one of the models $\mathfrak{N}_R$ determined (in the manner described above) by some congruence relation R of $\mathfrak{N}$. In view of Theorems IV and V we therefore see that every induction model is isomorphic to a model $\mathfrak{N}_R^*$ determined by some congruence relation R of the system $\mathfrak{N}^*$ of natural numbers.

As it happens, we can give an explicit description of all congruence relations on $\mathfrak{N}^*$ in terms of the familiar ordering relation $<$ on N^*.‡ Namely, let m, n be any elements of N^*. We define the relation $R_{m,n}$ on N^* by the rule that $xR_{m,n}y$ if and only if one of the following two conditions holds: (i) $x, y<n$ and $x=y$, (ii) $x, y\geqq n$ and for some $z\in N^*$ either $x=y+^*(z\cdot^* m)$ or $y=x+^*(z\cdot^* m)$.

THEOREM VII. *A binary relation R on N^* is a congruence relation of $\mathfrak{N}^*$ if and only if it is the identity relation on N^* or there exist numbers $m, n\in N^*$ such that $R=R_{m,n}$.*

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† An equivalence relation on N is a binary relation R, having N both for its domain and range, which is reflexive, symmetric, and transitive. It is an elementary fact of set-theory that for each such relation R there is a partition of N into disjoint subsets such that two elements are in the same subset if and only if these elements are in the relation R. The subset containing an element x is called the *equivalence set of x under R*; we will denote it x_R.

‡ This ordering relation can be introduced into the axiomatic theory of $\mathfrak{N}^*$ (*i.e.*, into the theory of Peano models), by the definition $x<y$ if and only if there is an element $z\neq 0^*$ such that $x+^*z=y$. Since every induction model possesses an addition function, we can use this definition to define a

We shall leave the proof of this theorem to the reader, contenting ourselves with giving the following hint. If R is a congruence relation other than the relation of identity, there must be an $x \in N^*$ such that xRy for some $y \neq x$. Choose n to be the least of these numbers x. It follows from this choice of n that there are numbers $z \neq 0$ such that $nRn+^*z$; choose m to be the least of these numbers z. It can then be shown that $R=R_{m,n}$.

The congruence relations $R_{m,0}$ are the so-called modular congruences which have been extensively studied in the theory of numbers.§ The induction model $\mathfrak{N}^*_{R_{m,0}}$ corresponding to such a relation is simply the system of residue classes modulo m, and the operations of addition and multiplication of this model are the familiar $+(\text{mod } m)$ and $\cdot$ (mod m). The congruence relations $R_{m,n}$ for $n \neq 0$ do not seem to have received much attention in the literature,# but a little reflection will suffice to give the reader a clear intuitive picture of the models $\mathfrak{N}^*_{R_{m,n}}$, as well as of the corresponding operations of addition and multiplication, in this case also.

Incidentally, it is evident that if R is one of the modular congruence relations $R_{m,0}$, then S^*_R is a permutation of the elements of N^*_R so that $\mathfrak{N}^*_R$ satisfies Axiom P2 in this case. On the other hand, if R is one of the congruence relations $R_{m,n}$ for $n \neq 0$, it is clear that $0^*_R \neq S^*_R x$ for all $x \in N_R$, so that $\mathfrak{N}^*_R$ satisfies Axiom P1 in this case. Thus every induction model satisfies either P1 or P2, as mentioned in Section 1.

Let f be any operation on $\mathfrak{N}^*$—let us say a binary operation for definiteness. If R is an arbitrary congruence relation on $\mathfrak{N}^*$, there is in general no binary operation g on N^*_R which is a homomorphic image of f under the homomorphism h which corresponds to R. It is not hard to see that the necessary and sufficient condition for the existence of such an image operation g is that for all $x, x_1, y, y_1 \in N^*$ such that xRx_1 and yRy_1 we have $(fxy)R(fx_1y_1)$. If this condition holds then g is the operation on N^*_R such that $gx_Ry_R=(fxy)_R$ for all $x, y \in N^*$.

For example, although $2 \equiv 2 \pmod 3$ and $0 \equiv 3 \pmod 3$ we have $2^0 \not\equiv 2^3 \pmod 3$. Hence the exponential operation of N^* has no homomorphic image in $N^*_{R_{3,0}}$. On the other hand, $+^*$ and $\cdot^*$ are examples of what we may call *universal operations* on N^*; that is, they are operations f with the property that for *any* congruence relation R of N^*, $(fxy)R(fx_1y_1)$ whenever x, y, x_1, y_1 are elements of N^* such that xRx_1 and yRy_1.† It is for this reason that every induction model possesses operations of addition and multiplication.

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relation $<$ in every ***induction*** model; but in general the relation so obtained will not be an ordering relation.

§ It is customary, in works on number-theory, to write $x \equiv y \pmod m$ instead of $xR_{m,0}y$.

A brief reference may be found in a note by H. S. Vandiver, Bull. Amer. Math. Soc. vol. 40, 1934, pp. 914–920.

† We may call f a ***modular operation*** if it has this property for all *modular* congruence relations R (and not necessarily for the other congruence relations). An interesting characterization of the modular operations has been given by N. G. de Bruijn. (Cf. Proc. Kon. Ned. Ak. Wetensch. Amsterdam, series A, 58 (Indagationes Math. 17), 1955, pp. 363–367).

Suppose that an operation j on N^* is obtained by primitive recursion from operations f and g which are universal; is j necessarily universal? The example of the exponential operation shows that this is not the case. However, if j happens to be commutative then it *will* be universal: this can be shown by a straightforward generalization of the proof of Theorem II. Thus it is because the exponential operation on N^* is noncommutative that we cannot extend the proof of Theorem II to show the existence of an exponential operation in every induction model.

Of course the condition of commutativity, although sufficient to guarantee universality under the stated conditions, is by no means necessary. For example, the operation j such that $jxy=x^2\cdot y$ for all $x, y\in N^*$ is a universal operation, and it is obtained by primitive recursion from the universal functions f, g such that $fx=0$ and $gxyz=z+x^2$ for all $x, y, z\in N^*$; but j is noncommutative.

7. A characterization of Peano models. In Section 2 we have seen that the justification for a definition by mathematical induction in a model $\mathfrak{N}$ is the existence of a unique homomorphism of $\mathfrak{N}$ into some other model; and thus Theorem I constitutes a justification of *all* definitions by mathematical induction in Peano models. As it happens, this property is characteristic for Peano models: these are the *only* models in which all definitions by mathematical induction are justified.

THEOREM VIII. *Let $\mathfrak{N}$ be a model such that, for any model $\mathfrak{N}_1$ there is a unique homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}_1$. Then $\mathfrak{N}$ is a Peano model.*

Proof. Let $\mathfrak{N}=\langle N, 0, S\rangle$, and suppose that $\mathfrak{N}$ satisfies the hypothesis of the theorem. We shall show first that $\mathfrak{N}$ satisfies Axiom P3. To this end let G be any subset of N which contains 0 and is closed under S. Let H be the complement of G (with respect to N), and assume that H is not empty. Let k be a one-one mapping of H onto a set P which is disjoint from N. Let M be the union of N and P. Define a unary operation T on M, as follows: If $x\in N$ then $Tx=Sx$; if $x\in H$ and $Sx\in H$ then $T(kx)=k(Sx)$; if $x\in H$ and $Sx\in G$ then $T(kx)=Sx$. Let $\mathfrak{M}$ be the model $\langle M, 0, T\rangle$. It is clear that the mapping h_1 of N into M such that $h_1x=x$ for all $x\in N$ is a homomorphism of $\mathfrak{N}$ into $\mathfrak{M}$. On the other hand, consider the following mapping h_2 of N into M: If $x\in G$ then $h_2x=x$; if $x\in H$ then $h_2x=kx$. It is not difficult to see that h_2 is also a homomorphism of $\mathfrak{N}$ into $\mathfrak{M}$, and that it is distinct from h_1. But this contradicts the hypothesis of our theorem, and hence shows that it was incorrect to assume H nonempty. H is empty, and so $G=N$. That is, $\mathfrak{N}$ must be an induction model.

From Theorem IV we can now infer that there is a homomorphism h' of $\mathfrak{N}^*$ onto $\mathfrak{N}$. On the other hand the hypothesis of our theorem assures us that there is a homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}^*$. As in the proof of Theorem V we consider the composed function (hh') which is a homomorphism of $\mathfrak{N}^*$ into itself, and so (by Theorem I) must be the identity operation on $\mathfrak{N}^*$. It follows that h' is one-one, and hence is an isomorphism of $\mathfrak{N}^*$ onto $\mathfrak{N}$. This proves the theorem.

It is clear from Theorem VIII that any proof of Theorem I must employ all of the Axioms P1–P3, as claimed in Section 2.

8. Conclusion. We have now finished our discussion of the concept of definition by mathematical induction in the theory of numbers, based upon Peano's axiomatic foundation for this theory. In the literature of mathematics there are various other types of inductive definitions employed, *e.g.*, definition by transfinite induction, or by induction in certain types of partially ordered systems. Many of the ideas of this paper can be generalized to cover these other types of induction.

While the method of presenting the material in this paper may have some claim to originality, several of the proofs which were given are well known to mathematicians. In particular, this is true of the two proofs (one given in detail, the other outlined) of Theorem I. According to Professor Alonzo Church, the origin of the first proof goes back to the Hungarian mathematician L. Kalmár, while the idea of the second proof should be credited to P. Lorenzen, and D. Hilbert and P. Bernays, who discovered the proof independently and published their work nearly simultaneously.† The proof of Theorem II is given by E. Landau‡ who credits it to L. Kalmár. However, Landau fails to note the significance of the fact that the proof does not use Axioms P1 and P2.§

† Kalmár's article appears in Acta Sci. Math. (Szeged), vol. 9, No. 4, 1950, pp. 227–232. Lorenzen's work appears in Monatsh. Math. und Phys., vol. 47, 1938–39, pp. 356–358, and the proof of Hilbert and Bernays appears in the Appendix to Volume 2 of their book Grundlagen der Mathematik.

‡ See E. Landau, Foundations of Analysis.

§ (Added October 2, 1958.) There has just come to my attention an article with several ideas closely related to those of this paper: H. Lenz, Zur Axiomatik der Zahlen, Acta Math. Acad. Sci. Hungar., vol. IX, 1958, pp. 33–44.

GLOBAL EQUILIBRIUM THEORY OF CHARGES ON A CIRCLE*

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1. Introduction. Recently electronic calculations were made [1] to see if there is some "brute force" solution to an old problem of finding equilibrium positions of charged particles constrained to lie on a sphere and acted upon by mutual Newtonian repulsion (see Föppl [2]). While some new stable minima seemed plausible on the basis of the calculation, the problem of verifying unstable equilibrium positions was too formidable for conclusive results from numerical data at present.

Surprisingly enough the analogous two-dimensional unstable equilibrium problem is far from trivial and yet can be analyzed completely; but it never seems to have made its way into the literature [6]. Here we would consider n

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