THE PENROSE INEQUALITY IN GENERAL RELATIVITY AND VOLUME COMPARISON THEOREMS INVOLVING SCALAR CURVATURE

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Abstract

In this thesis we describe how minimal surface techniques can be used to prove the Penrose inequality in general relativity for two classes of 3-manifolds. We also describe how a new volume comparison theorem involving scalar curvature for 3-manifolds follows from these same techniques.

The Penrose inequality in general relativity is closely related to the positive mass theorem, first proved by Schoen and Yau in 1979. In physical terms, the positive mass theorem states that an isolated gravitational system with nonnegative local energy density must have nonnegative total energy. The idea is that nonnegative energy densities "add up" to something nonnegative. The Penrose inequality, on the other hand, states that if an isolated gravitational system with nonnegative local energy density contains a black hole of mass m, then the total energy of the system must be at least m.

Given a 3-manifold M^3, we consider the function $A(V)$ equal to the minimum area required for a surface in M^3 to contain a volume V. We find that lower bounds on the curvature of M^3 yield upper bounds on $A''(V)$. Furthermore, in the case of an asymptotically flat manifold which has nonnegative scalar curvature (which is the condition needed for nonnegative energy density), we find that the behavior of $A(V)$ for large V describes the total mass of the manifold. In this way we are able to use the curvature bounds of the manifold to achieve lower bounds on the total mass. We can also use Ricci and scalar curvature bounds on a compact 3-manifold M^3 to bound the total volume of M^3. Since $A(V)$ equals zero when V is either equal to zero or the total volume of M^3, upper bounds on $A''(V)$ force the roots of $A(V)$ to be close together, giving an upper bound on the volume of M^3.

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Chapter 1

Introduction

Einstein's theory of general relativity is a theory of gravity which asserts that matter causes the four dimensional space-time in which we live to be curved, and that our perception of gravity is a consequence of this curvature. Let $(N^4, \bar{g})$ be the space-time manifold with metric $\bar{g}$ of signature $(- + ++)$. Then the central formula of general relativity is Einstein's equation,

$$G = 8\pi T, \tag{1.1}$$

where T is the energy-momentum tensor, $G = Ric(\bar{g}) - \frac{1}{2}R(\bar{g}) \cdot \bar{g}$ is the Einstein curvature tensor, $Ric(\bar{g})$ is the Ricci curvature tensor, and $R(\bar{g})$ is the scalar curvature of $\bar{g}$. The beauty of general relativity is that this simple formula explains gravity more accurately than Newtonian physics and is entirely consistent with large scale experiments.

However, the nature of the behavior of mass in general relativity is still not well understood. It is not even well understood how to define how much energy and momentum exists in a given region, except in special cases. There does exist a well defined notion of local energy and momentum density which is simply given by the energy-momentum tensor which, by equation 1.1, can be computed in terms of the curvature of N^4. Also, if we assume that the matter of the space-time manifold N^4 is concentrated in some central region of the universe, then N^4 becomes flatter as we get farther away from this central region. If the curvature of N^4 decays quickly enough, then N^4 is said to be asymptotically flat, so that with these assumptions it is then possible to define the total mass of the space-time N^4. Interestingly enough, though, the definition of local energy-momentum density, which involves curvature terms of N^4, bears no obvious resemblance to the definition of the total mass of N^4, which is defined in terms of how fast the metric becomes flat at infinity.

The Penrose inequality and the positive mass theorem can both be thought of as basic attempts at understanding the relationship between the local energy density of a space-time N^4 and the total mass of N^4. In physical terms, the positive mass theorem states that an isolated gravitational system with nonnegative local energy density must have nonnegative total energy. The idea is that nonnegative energy densities must "add up" to something nonnegative. The Penrose inequality, on the other hand, states that if an isolated gravitational system with nonnegative local energy density contains a black hole of mass m, then the total energy of the system must be at least m.

Important cases of the positive mass theorem and the Penrose inequality can be translated into statements about complete, asymptotically flat 3-manifolds (M^3, g) with nonnegative scalar curvature. If we consider (M^3, g) as a space-like hypersurface of $(N^4, \bar{g})$ with second fundamental form h_{ij} in N^4, then equation 1.1 implies that

$$\mu = \frac{1}{16\pi}[R - \sum_{i,j} h^{ij}h_{ij} + (\sum_i h^i_i)^2], \tag{1.2}$$

$$J^i = \frac{1}{8\pi}\sum_j \nabla_j[h^{ij} - (\sum_k h^k_k)g^{ij}], \tag{1.3}$$

where R is the scalar curvature of the metric g, μ is the local energy density, and J^i is the local current density. These two equations are called the constraint equations for M^3 in N^4, and the assumption of nonnegative energy density everywhere in N^4 implies that we must have

$$\mu \geq \left(\sum_i J^i J_i\right)^{\frac{1}{2}} \tag{1.4}$$

at all points on M^3 [28]. Thus we see that if we restrict our attention to 3-manifolds which have zero mean curvature in N^3, the constraint equations and the assumption of nonnegative energy density imply that (M^3, g) has nonnegative scalar curvature everywhere. We also assume that (M^3, g) is asymptotically flat, which is defined in section 2.4, in which case we can define the total mass of M^3, also given in section 2.4.

An "end" of an n-manifold is a region of the manifold diffeomorphic to $\mathbf{R}^n - B_1(0)$ where $B_1(0)$ is the ball of radius one in $\mathbf{R}^n$. In general, M^3 may have any number of disjoint ends, but for simplicity, let us assume that M^3 has only one disjoint end and that it is asymptotically flat. In section 2.4 we will show that without loss of generality (for stating the Penrose inequality and the positive mass theorem) we may assume that $(M^3 - K, g)$ is isometric to $(\mathbf{R}^3 - B, h)$ for some compact set K in M^3 and some ball B in $\mathbf{R}^3$ centered around the origin, and for some constant m, where $h_{ij} = (1+\frac{m}{2r})^4\delta_{ij}$ and r is the radial coordinate in $\mathbf{R}^3$. This is a convenient assumption about M^3, because the total mass of the system is then just m. The metric $(\mathbf{R}^3 - \{0\}, h)$ has zero scalar curvature, is spherically symmetric, and is called the Schwarzschild metric of mass m, and we say that in the above case, M^3 is Schwarzschild with mass m at infinity. Using this simplified setup, we can make a statement which is equivalent to the positive mass theorem in this setting.

The Positive Mass Theorem (Schoen, Yau, 1979) *Suppose (M^3, g) is complete, has nonnegative scalar curvature, and is Schwarzschild with mass m at infinity. Then $m \geq 0$, and $m = 0$ if and only if (M^3, g) is isometric to $\mathbf{R}^3$ with the standard flat metric.*

Apparent horizons of black holes in N^4 correspond to outermost minimal spheres of M^3 if we assume M^3 has zero second fundamental form in N^4. An outermost minimal sphere is a sphere in M^3 which locally minimizes area (and hence has zero mean curvature) and which is not contained entirely inside another minimal sphere. We will also use the term horizon to mean an outermost minimal sphere in M^3. It is easy to show that two outermost horizons never intersect. Also, it follows from a stability argument that these minimal surfaces are always spheres [12]. However, there may be more than one outermost minimal sphere, with each minimal sphere corresponding to a different black hole. As we will see in the next section, there is a strong motivation to define the mass of a black hole as $\sqrt{\frac{A}{16\pi}}$, where A is the surface area of the horizon. Hence, the physical statement that a system with nonnegative energy density containing a black hole of mass m must have total mass at least m can be translated into the following geometric statement.

The Penrose Inequality (Huisken, Ilmanen, announced 1997) *Suppose* (M^3, g) *is complete, has nonnegative scalar curvature, contains an outermost minimal sphere with surface area* A*, and is Schwarzschild with mass* m *at infinity. Then* $m \geq \sqrt{\frac{A}{16\pi}}$*, with equality only in the case that* (M^3, g) *is isometric to the Schwarzschild metric of mass* m *outside the horizon.*

When $m > 0$, the Schwarzschild metrics $(\mathbf{R}^3, h)$ have minimal spheres at $r = \frac{m}{2}$ with areas $16\pi m^2$ so that these metrics give equality in the Penrose inequality, and in fact, according to the recent announcement of Huisken and Ilmanen, these are the only metrics (outside the horizon) which give equality.

The proof that Huisken and Ilmanen used to prove the Penrose inequality is as interesting as the theorem itself. We discuss the main ideas of their proof in section 1.4. We also give another proof of the Penrose inequality for two classes of manifolds in chapter 2 using isoperimetric surface techniques. Both approaches are also interesting because they give hints about the nature of quasi-local mass in general relativity.

Also, using isoperimetric surface techniques we are able to prove a generalized Penrose inequality for a class of manifolds in the case that (M^3, g) has more than one horizon. The idea is that if (M^3, g) has more than one horizon, then it should be possible to bound the total mass from below by some function of the areas of the horizons. In this way we hope to understand how masses "add" in general relativity. We state the conditions under which we can prove a generalized Penrose inequality in the introduction to chapter 2 and then conjecture that this generalized Penrose inequality is always true.

1.1 Motivation behind the Penrose Inequality

In 1973, Roger Penrose proposed the Penrose inequality as a test of the cosmic censor hypothesis [23]. The cosmic censor hypothesis states that naked singularities do not develop starting with physically reasonable nonsingular generic initial conditions for the Cauchy problem in general relativity. (However, it has been shown by Christodoulou [5] that naked singularities can develop from nongeneric initial conditions.) If naked singularities did typically develop from generic initial conditions, then this would be a serious problem for general relativity since it would not be possible to solve the Einstein equations uniquely past these singularities. Singularities such as black holes do develop but are shielded from observers at infinity by their horizons so that the Einstein equations can still be solved from the point of view of an observer at infinity.

A summary of Penrose's argument can be found in [22]. The main idea is to consider a space-time $(N^4, \bar{g})$ with given initial conditions for the Cauchy problem (M^3, g) with zero second fundamental form in N^4. We assume that N^4 has nonnegative energy density everywhere, so by the constraint equations M^3 must have nonnegative scalar curvature. Suppose also (M^3, g) has an outermost apparent horizon of area A, and event horizon of area A_i, and total mass m_i (see [12], [13] for the definitions of these horizons). As long as a singularity does not form, then it is assumed that eventually the space-time should converge on some stationary final state. From the theorems of Israel [17], Hawking [11], and Robinson [24], the only stationary vacuum black holes are the Kerr solutions which satisfy

$$A_f = 8\pi[m_f^2 + (m_f^4 - J^2)^{\frac{1}{2}}] \le 16\pi m_f^2, \tag{1.5}$$

where A_f is the area of the horizon of the Kerr black hole, m_f is the mass at infinity, and J is the angular momentum.

However, by the Hawing area theorem [10], the area of the event horizon of the black hole is nondecreasing. Thus, $A_f \ge A_i$. Also, presumably some energy radiates off to infinity, so we expect to have $m_i \ge m_f$.

The apparent horizon is defined to be the outer boundary of the region in M^3

which contains trapped or marginally trapped surfaces [12]. The apparent horizon itself must then be a marginally trapped surface, and hence satisfies

$$H + h^{ij}(g_{ij} - r_i r_j) = 0 \tag{1.6}$$

where H is the mean curvature of the apparent horizon in M^3, h is the second fundamental form of (M^3, g) in $(N^4, \bar{g})$, and r is the outward unit normal to the apparent horizon in M^3. Hence, since we chose M^3 to have zero second fundamental form, $h^{ij} = 0$, so that the apparent horizon is a zero mean curvature surface in M^3. Furthermore, if we consider the surface of smallest area which encloses the apparent horizon, it too must have zero mean curvature and hence is a marginally trapped surface in M^3. Thus, the apparent horizon is an outermost minimal surface of M^3, which by stability arguments, must be a sphere [12]. Since the event horizon always contains the apparent horizon, $A_i \geq A$, so putting all the inequalities together we conclude that

$$m_i \geq m_f \geq \sqrt{\frac{A_f}{16\pi}} \geq \sqrt{\frac{A_i}{16\pi}} \geq \sqrt{\frac{A}{16\pi}} \tag{1.7}$$

Thus, Penrose argued, assuming the cosmic censor hypothesis and a few reasonable sounding assumptions as to the nature of gravitational collapse, given a complete asymptotically flat 3-manifold M^3 of total mass m_i with nonnegative scalar curvature which has an outermost minimal sphere of total area A, then

$$m_i \geq \sqrt{\frac{A}{16\pi}} \tag{1.8}$$

Conversely, he argued, if one could find an M^3 which was a counterexample to the above inequality, then it would be likely that the counterexample, when used as initial conditions in the Cauchy problem for Einstein's equation, would produce a naked singularity. Since Huisken and Ilmanen have proved the above inequality, they have ruled out one possible way of constructing counterexamples to the cosmic censor hypothesis.

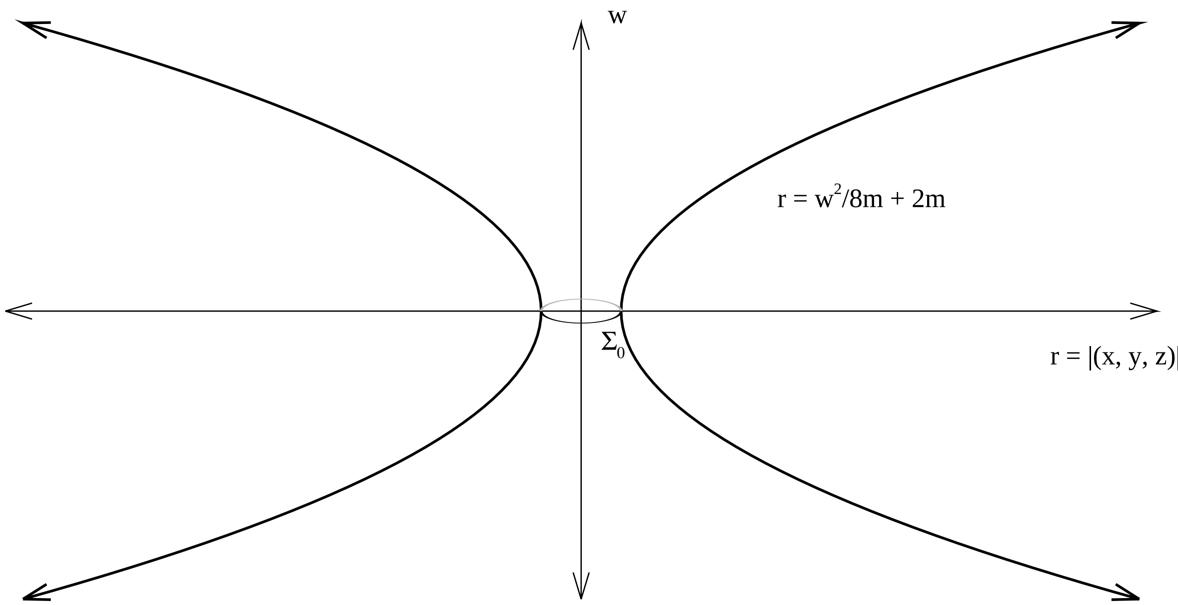

Figure 1.1: The space-like Schwarzschild metric of mass m, $(\mathbf{R}^3 - \{0\}, h)$, viewed as a submanifold of four-dimensional Euclidean space.

1.2 The Schwarzschild Metric

The space-like Schwarzschild metric is a particularly important example to consider when discussing the Penrose inequality. First of all, Huisken and Ilmanen prove that it is the only 3-manifold which gives equality in the Penrose inequality. Also, if a 3-manifold is assumed to be complete, spherically symmetric, and have zero scalar curvature, then it must be isometric to either a Schwarzschild metric of mass $m > 0$ or $\mathbf{R}^3$, which can be viewed as the Schwarzschild metric when $m = 0$.

In addition, understanding the Schwarzschild metric is particularly important for chapter 2 because we show in section 2.4 that without loss of generality for proving the Penrose inequality for M^3 we may assume that outside a compact set M^3 is spherically symmetric with zero scalar curvature, which means that in this region it is isometric to the Schwarzschild metric of some mass m. We also show in that same section that all asymptotically flat metrics of nonnegative scalar curvature can be perturbed pointwise less than ϵ in such a way that the total mass is changed less than

ϵ too and so that the new metric is isometric to the Schwarzschild metric outside a compact set. Thus, the Schwarzschild metric is a useful picture to keep in mind.

The space-like Schwarzschild metric, $(\mathbf{R}^3-\{0\},h)$, is a time symmetric asymptotically flat three-dimensional maximal slice (chosen to have zero momentum at infinity) of the four-dimensional Schwarzschild space-time metric.The space-like Schwarzschild metric is conformal to $\mathbf{R}^3-\{0\}$ with $h_{ij}=\left(1+\frac{m}{2r}\right)^4\delta_{ij}$. The Schwarzschild metric of mass m, $(\mathbf{R}^3-\{0\},h)$, can also be isometrically embedded into four-dimensional Euclidean space as the three-dimensional set of points in $\mathbf{R}^4=\{(x,y,z,w)\}$ satisfying $|(x,y,z)|=\frac{w^2}{8m}+2m$, seen in figure 1.1. Hence, Σ_0 is a minimal sphere of area $16\pi m^2$, so we have equality in the Penrose inequality.

1.3 The Spherically Symmetric Case

In this section we sketch a proof of the Penrose inequality in the case that M^3 is spherically symmetric. The proof is very easy conceptually, but what is more important is that some of the ideas generalize. In particular, we will see why the minimal sphere in the Penrose inequality must be outermost.

Let (M^3,g) be a complete asymptotically flat spherically symmetric 3-manifold with nonnegative scalar curvature. For convenience, we also assume that (M^3,g) is isometric to the Schwarzschild metric of some mass m outside a large compact set. Then the total mass of (M^3,g) is m. Let $\Sigma(V)$ be the spherically symmetric sphere containing a volume V in M^3. Let $A(V)$ be the area of this sphere. It turns out that the function $A(V)$, $V\geq 0$, captures all the information about M^3 since M^3 is spherically symmetric.

Let $R(V)$ be the scalar curvature of M^3 on $\Sigma(V)$. From the calculations we will do in section 2.1, it follows that

$$R(V)=\frac{8\pi}{A}-2A(V)A''(V)-\frac{3}{2}A'(V)^2 \tag{1.9}$$

Define

$$m(V)=\left(\frac{A(V)}{16\pi}\right)^{\frac{1}{2}}\left(1-\frac{1}{16\pi}A(V)A'(V)^2\right) \tag{1.10}$$

It turns out that $m'(V) \geq 0$ whenever $A'(V) \geq 0$ since we find that

$$m'(V) = \frac{A'(V)}{16\pi} \left(\frac{A(V)}{16\pi} \right)^{\frac{1}{2}} R(V) \tag{1.11}$$

and $R(V) \geq 0$.

Let $\Sigma(V_0)$ be the outermost minimal sphere. It follows that $A'(V) \geq 0$ for all $V \geq V_0$. Hence, $m(V)$ is increasing in this range as well, so

$$\lim_{V \to \infty} m(V) \geq m(V_0) \tag{1.12}$$

Furthermore $m(V_0) = \sqrt{\frac{A(V_0)}{16\pi}}$ since $A'(V_0) = 0$. Also, we assumed that M^3 was isometric to the Schwarzschild metric outside a large compact set, and we claim that $m(V) = m$, the mass parameter of the Schwarzschild metric, in this region, or equivalently, for $V > V_{LARGE}$ for some $V_{LARGE} > 0$. To see this, consider the mass function $m(V)$ defined on the Schwarzschild metric, where now V refers to the volume contained by the spherically symmetric spheres of the Schwarzschild metric which is outside the horizon. Then by equation 1.11, $m(V)$ is constant for all V on the Schwarzschild metric since the Schwarzschild metric has zero scalar curvature. Furthermore, setting $V = 0$ and considering $m(V)$ at the horizon yields $m(0) = \sqrt{\frac{A(0)}{16\pi}} = m$, the mass parameter of the Schwarzschild metric, since the Schwarzschild metric gives equality in the Penrose inequality. Thus, $m(V) = m$ for all V in the Schwarzschild metric, so going back to (M^3, g), we see that $m(V) = m$, the mass parameter of the Schwarzschild metric, for $V > V_{LARGE}$. Thus, it follows from inequality 1.12 that

$$m \geq \sqrt{\frac{A(V_0)}{16\pi}} \tag{1.13}$$

which proves the Penrose inequality for spherically symmetric manifolds.

Conversely, we see that equation 1.11 can be used to construct spherically symmetric manifolds which do not satisfy the Penrose inequality if we do not require the minimal sphere to be outermost. In figure 1.2, we are viewing (M^3, g) as an isometrically embedded submanifold of $\mathbf{R}^4$ with the standard Euclidean metric. (M^3, g) is

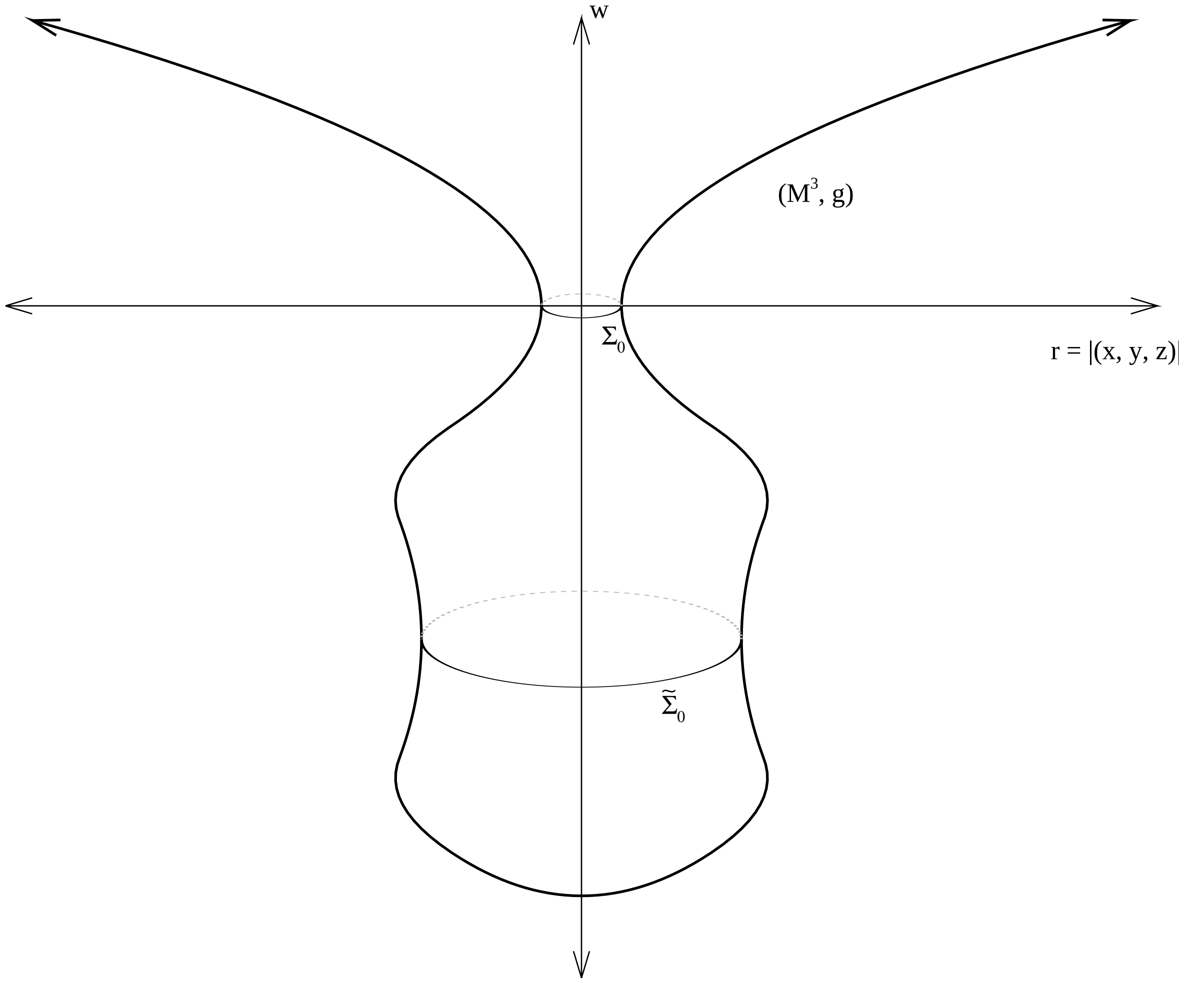

Figure 1.2: Counterexample to Penrose inequality if the minimal sphere is not outermost.

spherically symmetric and is constructed by rotating the curve shown above around the w-axis in $\mathbf{R}^4$. Hence, Σ_0 and $\tilde{\Sigma}_0$ are both 2-spheres, and we can choose the curve shown above so that the scalar curvature of (M^3, g), $R(g)$, is non-negative.

The Penrose inequality, $m \geq \sqrt{\frac{|\Sigma_0|}{16\pi}}$, is true for Σ_0, but is not true for $\tilde{\Sigma}_0$. However, $\tilde{\Sigma}_0$ is not an outermost minimal sphere since it is contained by another minimal sphere, namely, Σ_0. In fact, since $\tilde{\Sigma}_0$ is not outermost, we can construct a spherically symmetric manifold like the one shown above so that the area of $\tilde{\Sigma}_0$ is as large as we like and the total mass of (M^3, g) is still one.

1.4 The Hawking Mass and Inverse Mean Curvature Flows

One goal in general relativity is to understand how to define the amount of mass inside a given region. In section 1.3, we defined a function $m(V)$ which was increasing as a function of V outside the outermost minimal sphere. Furthermore, for large V, $m(V)$ equaled the total mass of the manifold. Hence, it seems reasonable to say that the spherically symmetric sphere $\Sigma(V)$ defined in section 1.3 contains a mass $m(V)$. The function $m(V)$ is called a quasi-local mass function.

Naturally we would like to define a quasi-local mass function which would measure the amount of mass inside any surface Σ which is the boundary of a region in any 3-manifold M^3. We refer the reader to [6], [3] ,[7], and [9] for a complete discussion of this topic.

In this section we discuss a definition of quasi-local mass proposed by Hawking called the Hawking mass. Going back to section 1.3, we recall that in the spherically symmetric case,

$$m(V) = \left(\frac{A(V)}{16\pi}\right)^{\frac{1}{2}} \left(1 - \frac{1}{16\pi} A(V) A'(V)^2\right) \tag{1.14}$$

As can be seen from the calculations in the appendix and as will be shown later, it so happens that $A'(V) = H(V)$, where $H(V)$ is the mean curvature of the spherically symmetric sphere $\Sigma(V)$. Hence, one way to generalize equation 1.14 is to define

$$m(\Sigma) = \left(\frac{A}{16\pi}\right)^{\frac{1}{2}} \left(1 - \frac{1}{16\pi}\int_\Sigma H^2\right) \tag{1.15}$$

where A is the area of Σ and H is the mean curvature of Σ in M^3. It turns out that this definition of quasi-local mass has some very important properties.

In [7], Geroch showed that if M^3 has nonnegative scalar curvature, then the Hawking mass is nondecreasing when the surface Σ is flowed out at a rate equal to the inverse of its mean curvature. This is straight forward to check using equations A.2 and A.3 from the appendix and the Gauss equation which is given in equation 2.1 of section 2.1. In view of this, Jang and Wald proposed using the Hawking mass function to prove the Penrose inequality [22]. They suggested that we should let $\Sigma(0)$ be an outermost horizon and then to flow out using an inverse mean curvature flow to create a family of surfaces $\Sigma(t)$ flowing out to infinity. Since the Hawking mass function $m(t) = m(\Sigma(t))$ is nondecreasing as a function of t, we have

$$\lim_{t\to\infty} m(t) \geq m(0) \tag{1.16}$$

Furthermore $m(0) = \sqrt{\frac{A}{16\pi}}$, where A is the area of the outermost horizon $\Sigma(0)$, since horizons have zero mean curvature. Hence, Jang and Wald proposed a proof of the Penrose inequality which is basically a generalization of the proof which works in the spherically symmetric case.

The main problem for this type of proof is the existence of an inverse mean curvature flow. Naturally, if the mean curvature of the surface ever went to zero or became negative, the flow could not exist, at least in this form. However, Huisken and Ilmanen have recently announced that they have been able to generalize the idea of an inverse mean curvature flow to a "weak" inverse mean curvature flow which always exists and hence can be used to prove the Penrose inequality [15].

They introduce the notion of a "maximal minimal mean convex hull" of a surface Σ which equals the outermost surface of minimum area needed to enclose Σ. Then their weak inverse mean curvature flow can be thought of as continuously replacing

$\Sigma(t)$ with the maximal minimal mean convex hull of $\Sigma(t)$ while flowing out using the inverse mean curvature flow. The replacement process can then be shown to never decrease the mass and also to keep the mean curvature of $\Sigma(t)$ nonnegative. The resulting weak flow is a family of surfaces which occasionally has "jumps" and for which the Hawking mass is nondecreasing. They use this to prove the Penrose inequality as stated in the introduction to this chapter.

Partial results on the Penrose inequality have also been found by Herzlich [14] using the Dirac operator which Witten [33] used to prove the positive mass theorem, by Gibbons [8] in the special case of collapsing shells, by Tod [32], and by Bartnik [4] for quasi-spherical metrics.

However, other versions of the Penrose inequality remain open. As in the positive mass theorem [29], the assumption of nonnegative scalar curvature for M^3 should be able to be modified to include a more general local nonnegative energy condition. It is also natural to ask what kind of generalized Penrose inequality we should expect for manifolds with multiple horizons. In chapter 2 we prove a generalized Penrose inequality for a certain class of manifolds and in section 2.10 conjecture that this generalized inequality is always true.

1.5 Volume Comparison Theorems

The isoperimetric surface techniques which we will develop to study the Penrose inequality in general relativity also can be used to prove several volume comparison theorems, including a new proof of Bishop's volume comparison theorem for positive Ricci curvature.

Let (S^n, g_0) be the standard metric (with any scaling) on S^n with constant Ricci curvature $Ric_0 \cdot g_0$. Bishop's theorem says that if (M^n, g) is a complete Riemannian manifold ($n \geq 2$) with $Ric(g) \geq Ric_0 \cdot g$, then $\mathrm{Vol}(M^n) \leq \mathrm{Vol}(S^n)$. It is then natural to ask whether a similar type of volume comparison theorem could be true for scalar curvature. As it happens, a lower bound on scalar curvature by itself is not sufficient to give an upper bound on the total volume. We can scale a cylinder, $S^2 \times \mathbf{R}$, to have any positive scalar curvature and still have infinite volume.

Since a lower bound on scalar curvature is not enough to realize an upper bound on the volume of a manifold, in chapter 3 we consider 3-manifolds (M^3, g) which satisfy $R(g) \geq R_0$ and $Ric(g) \geq \epsilon \cdot Ric_0 \cdot g$, where (S^3, g_0) is the standard metric on S^3 with constant scalar curvature R_0 and constant Ricci curvature $Ric_0 \cdot g_0$ (so that naturally $R_0 = 3Ric_0$). It turns out that there do exist values of $\epsilon < 1$ for which these curvature conditions imply that $\mathrm{Vol}(M^3) \leq \mathrm{Vol}(S^3)$, giving us a volume comparison theorem for scalar curvature. We also find the best value for ϵ for which this theorem is true.

Chapter 2

The Penrose Inequality

We will use isoperimetric surfaces to prove the Penrose inequality for two cases. In the first case, we will use surfaces which globally minimize area among surfaces which contain the same volume, and in the second case, we will need to look at collections of surfaces, each of which locally minimizes area among surfaces enclosing the same volume. In both cases these surfaces will have constant mean curvature.

Definition 1 *Suppose M^3 is asymptotically flat, complete, and has only one outermost minimal sphere Σ_0. Let $\tilde{M}^3$ be the closure of the component of $M^3 - \Sigma_0$ that contains the asymptotically flat end. We define*

$$A(V) = \inf_{\Sigma}\{\mathit{Area}(\Sigma) \mid \Sigma \textit{ contains a volume } V \textit{ outside } \Sigma_0\}$$

where Σ is the boundary of some 3-dimensional region in M^3 and Σ is a surface in $\tilde{M}^3$ in the same homology class of $\tilde{M}^3$ as the horizon Σ_0.

If Σ contains a volume V outside the horizon and $\text{Area}(\Sigma) = A(V)$, then we say that Σ minimizes area with the given volume constraint. Naturally, Σ could have multiple components, as long as one of the components contains the horizon.

Condition 1 (M^3, g) *has only one horizon Σ_0, and for each $V > 0$, if one or more area minimizers exist for V, then at least one of these area minimizers for the volume V has exactly one component.*

Note that condition 1 does not assume that an area minimizer exists for each V, just that if one or more do exist for V, at least one of these minimizers for V has only one component. However, once condition 1 is assumed, the existence of an area minimizer $\Sigma(V)$ for each $V \geq 0$ follows from the behavior of $A(V)$ as will be shown in section 2.7. Also, assuming condition 1, we can prove the Penrose inequality.

Theorem 1 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, contains an outermost minimal sphere with surface area A, is Schwarzschild with mass m at infinity, and satisfies condition 1. Then $m \geq \sqrt{\frac{A}{16\pi}}$.*

Naturally, we want to find a way to get around condition 1. This can be partially accomplished if we consider a different minimization problem, minimizing the sum of the areas to the three halves power given a volume constraint. Posing this type of problem seems strange at first, but turns out to be surprisingly natural for manifolds with nonnegative scalar curvature.

Definition 2 *Suppose M^3 is asymptotically flat, complete, and has any number of horizons. Let $\tilde{M}^3$ be the closure of the component of $M^3 - \{$the horizons$\}$ that contains the asymptotically flat end. Let*

$$F(V) = \inf_{\{\Sigma_i\}} \{\sum_i Area(\Sigma_i)^{\frac{3}{2}} \mid \{\Sigma_i\} \text{ contain a volume } V \text{ outside the horizons}\}$$

where the $\{\Sigma_i\}$ are the boundaries of the components of some 3-dimensional open region in M^3 and $\bigcup_i \Sigma_i$ is in $\tilde{M}^3$ and is in the homology class of $\tilde{M}^3$ which contains both a large sphere at infinity and the union of the horizons.

If the collection $\{\Sigma_i\}$ contains a volume V outside the horizons and $\sum_i \text{Area}(\Sigma_i)^{\frac{3}{2}} = F(V)$, then we say that $\{\Sigma_i\}$ minimizes F for the volume V. The only problem that occurs with this optimization problem is that two or more surfaces Σ_i and Σ_j can push up against each other.

Condition 2 *For each $V > 0$, if one or more sets of surfaces minimize F for the volume V, then at least one of these sets of surfaces $\{\Sigma_i\}$ which minimize F for the volume V is pairwise disjoint, that is, $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$.*

Note that condition 2 does not assume that an F minimizer exists for each V, just that if one or more do exist for V, at least one of these F minimizers for V does not have any of its surfaces pushing up against or touching each other. However, once condition 2 is assumed, the existence of an F minimizer for each $V \geq 0$ follows in a nice way from the behavior of $F(V)$ as we will see in section 2.8. Also, it is possible to verify experimentally using a computer to construct axially symmetric, conformally flat metrics with multiple horizons that there are examples of 3-manifolds which appear to satisfy condition 2 but not condition 1.

At first glance the issue of existence for this optimization problem looks bleak for several reasons. First, two components Σ_i and Σ_j can be joined into one component by a thread of area zero. This is always disadvantageous for minimizing F and so is not a problem. Also, one component of $\{\Sigma_i\}$ might run off to infinity. In addition, "bubbling" might occur, where the optimal configuration is an infinite number of tiny balls with a finite total volume. Amazingly, if we assume condition 2, these last two problems do not occur and the Penrose inequality follows.

Theorem 2 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, contains one or more outermost minimal spheres with surface areas $\{A_i\}$, is Schwarzschild with mass m at infinity, and satisfies condition 2. Then $m \geq \left(\sum_{i=1}^n \left(\frac{A_i}{16\pi}\right)^{\frac{3}{2}}\right)^{\frac{1}{3}}$.*

Thus, condition 2 implies a stronger version of the Penrose inequality since $\left(\sum_{i=1}^n \left(\frac{A_i}{16\pi}\right)^{\frac{3}{2}}\right)^{\frac{1}{3}} \geq \sqrt{\frac{A_j}{16\pi}}$ for all j. Based on this, it seems plausible to conjecture that for multiple black holes, this stronger Penrose inequality is always true. We provide additional motivation for this conjecture in section 2.10.

Conjecture 1 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, contains one of more outermost minimal spheres with surface areas $\{A_i\}$, and is Schwarzschild with mass m at infinity. Then $m \geq \left(\sum_{i=1}^n \left(\frac{A_i}{16\pi}\right)^{\frac{3}{2}}\right)^{\frac{1}{3}}$.*

2.1 Isoperimetric Surface Techniques

We will prove theorems 1 and 2 using constant mean curvature surfaces which minimize area given a volume constraint. First, let us assume the hypotheses of theorem 1 including condition 1 and recall the definition of $A(V)$ given in the previous section. Under these circumstances, we show in section 2.7 that for all $V \geq 0$, there exists a smooth, constant mean curvature surface $\Sigma(V)$ which minimizes area among surfaces which enclose a volume V outside the horizon. By condition 1, we may choose the minimizer $\Sigma(V)$ to have only one component, and since $\Sigma(V)$ is a minimizer, the area of $\Sigma(V)$ is $A(V)$.

$A(V)$ contains important geometric information, including the fact that $A(0)$ is the area of the horizon. The fact that the horizon is outermost implies that $A(V)$ is nondecreasing.

We must use this last fact somewhere, because the Penrose inequality is definitely not true without the assumption that the minimal sphere in the conjecture is outermost. In fact, it is easy to construct a complete, spherically symmetric 3-manifold with nonnegative scalar curvature and total mass 1 and an arbitrarily large (non-outermost) minimal sphere. It is worth noting that when considering other possible approaches to the Penrose inequality, the hypothesis of the Penrose inequality that the minimal sphere is outermost is often one of the more delicate and difficult points to handle.

Also, as we will prove in section 2.6, the total mass m of (M^3, g), is encoded in the asymptotic behavior of the function $A(V)$ for large V. Hence, the key to proving theorem 1 is understanding how the assumption of nonnegative scalar curvature on (M^3, g) bounds the behavior of $A(V)$.

Theorem 3 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, is Schwarzschild at infinity, and satisfies condition 1. Then the function $A(V)$ defined in definition 1 satisfies*

$$A''(V) \leq \frac{4\pi}{A(V)^2} - \frac{3A'(V)^2}{4A(V)}$$

in the sense of comparison functions, where this means that for all $V_0 \geq 0$ there exists

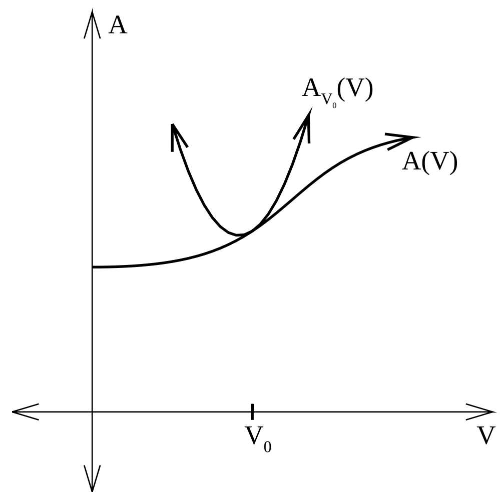

Figure 2.1: Graphical demonstration that $A''(V_0) \le A''_{V_0}(V_0)$.

a smooth function $A_{V_0}(V) \ge A(V)$ *with* $A_{V_0}(V_0) = A(V_0)$ *satisfying*

$$A''_{V_0}(V_0) \le \frac{4\pi}{A_{V_0}(V_0)^2} - \frac{3A'_{V_0}(V_0)^2}{4A_{V_0}(V_0)}$$

Proof. First we comment that all the inequalities which we state "in the sense of comparison function" are also true distributionally. We do not need this, so we do not prove it, but the proof is very similar to the proof of lemma 1 in section 2.2.

To get an upper bound for $A''(V)$ at $V = V_0$, we will do a unit normal variation on $\Sigma(V_0)$. Let $\Sigma_{V_0}(t)$ be the surface created by flowing $\Sigma(V_0)$ out at every point in the normal direction at unit speed for time t. Since $\Sigma(V_0)$ is smooth, we can do this variation for $t \in (-\delta, \delta)$ for some $\delta > 0$. Abusing notation slightly, we can also parameterize these surfaces by their volumes as $\Sigma_{V_0}(V)$ so that $V = V_0$ corresponds to $t = 0$. Let $A_{V_0}(V) = \text{Area}(\Sigma_{V_0}(V))$. Then $A(V_0) = A_{V_0}(V_0)$ and $A(V) \le A_{V_0}(V)$ since $\Sigma_{V_0}(V)$ is not necessarily minimizing for its volume. Hence,

$$A''(V_0) \le A''_{V_0}(V_0).$$

To compute $A''_{V_0}(V_0)$, we will need to compute the first and second derivatives of the area of $\Sigma_{V_0}(t)$ and the volume that it encloses. We will use the formulas

$$\dot{d\mu} = H\,d\mu \quad \text{and} \quad \dot{H} = -||\Pi||^2 - Ric(\nu,\nu)$$

where the dot represents differentiation with respect to t, $d\mu$ is the surface area 2-form for $\Sigma_{V_0}(t)$, Π is the second fundamental form for $\Sigma_{V_0}(t)$, $H = \text{trace}(\Pi)$ is the mean curvature, and ν is the outward pointing unit normal vector. Since $A_{V_0}(t) = \int_{\Sigma_{V_0}(t)} d\mu$,

$$A'_{V_0}(t) = \int_{\Sigma_{V_0}(t)} H\,d\mu$$

And since $V'(t) = \int_{\Sigma_{V_0}(t)} d\mu = A_{V_0}(t)$, we have that at $t = 0$,

$$A'_{V_0}(V) = A'_{V_0}(t)/V'(t) = H$$

By single variable calculus,

$$A''_{V_0}(V) = \frac{A''_{V_0}(t) - A'_{V_0}(V)V''(t)}{V'(t)^2}$$

so that at $t = 0$,

$$\begin{aligned} A_{V_0}(V_0)^2 A''_{V_0}(V_0) &= A''_{V_0}(t) - HV''(t) \\ &= \frac{d}{dt}\int_{\Sigma_{V_0}(t)} H\,d\mu \;-\; H\frac{d}{dt}\int_{\Sigma_{V_0}(t)} d\mu \\ &= \int_{\Sigma(V_0)} \dot{H}\,d\mu \\ &= \int_{\Sigma(V_0)} -||\Pi||^2 - Ric(\nu,\nu) \end{aligned}$$

By the Gauss equation,

$$Ric(\nu,\nu) = \frac{1}{2}R - K + \frac{1}{2}H^2 - \frac{1}{2}||\Pi||^2 \tag{2.1}$$

where R is the scalar curvature of M^3 and K is the Gauss curvature of $\Sigma(V_0)$. Substituting we get,

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} -\frac{1}{2}R + K - \frac{1}{2}H^2 - \frac{1}{2}||\Pi||^2$$

Since $\Sigma(V_0)$ has only one component, $\int_{\Sigma(V_0)} K = 2\pi\chi(\Sigma(V_0)) \leq 4\pi$ by the Gauss-Bonnet theorem. Since $R \geq 0$ and $||\Pi||^2 \geq \frac{1}{2}H^2$, we have

$$\begin{aligned} A_{V_0}(V_0)^2 A''_{V_0}(V_0) &\leq 4\pi - \int_{\Sigma(V_0)} \frac{3}{4}H^2 \\ &= 4\pi - \frac{3}{4}H^2 A_{V_0}(V_0) \end{aligned}$$

Hence,

$$A''_{V_0}(V_0) \leq \frac{4\pi}{A_{V_0}(V_0)^2} - \frac{3A'_{V_0}(V_0)^2}{4A_{V_0}(V_0)}$$

Finally, since $A(V_0) = A_{V_0}(V_0)$ and $A(V) \leq A_{V_0}(V)$ for every $V_0 \geq 0$,

$$A''(V) \leq \frac{4\pi}{A(V)^2} - \frac{3A'(V)^2}{4A(V)} \tag{2.2}$$

in the sense of comparison functions. □

It turns out that $\tilde{F}(V) = A(V)^{\frac{3}{2}}$ is more convenient to work with than $A(V)$. Note that $\tilde{F}$ and V have the same units. Making this substitution, inequality 2.2 becomes

$$\tilde{F}''(V) \leq \frac{36\pi - \tilde{F}'(V)^2}{6\tilde{F}(V)} \tag{2.3}$$

in the sense of comparison functions. This last inequality will be the key step in proving theorem 1.

Now we turn to the other case in which we can prove the Penrose conjecture where there may be multiple horizons. We assume the hypotheses of theorem 2 including condition 2 and recall the definition of $F(V)$ given in the previous section. Under these circumstances, we show in section 2.8 that for all $V \geq 0$, there exists a collection $\{\Sigma_i(V)\}$ of smooth surfaces which minimizes F among collections of surfaces which enclose a volume V outside the horizons. The mean curvature is constant (but generally different) on each component. By condition 2, we may take $\{\Sigma_i(V)\}$ to be pairwise disjoint, and since $\{\Sigma_i(V)\}$ is a minimizer, $\sum_i \text{Area}(\Sigma_i(V))^{\frac{3}{2}} = F(V)$.

$F(V)$ also contains important geometric information, including the fact that $F(0)$ is the sum of the areas of the horizons to the three halves power. The fact that the horizons are outermost implies that $F(V)$ is nondecreasing.

Again, as is proved in section 2.6, the total mass m of (M^3, g), is encoded in the asymptotic behavior of the function $F(V)$ for large V, since we will show that for sufficiently large V, the minimizing collection of surfaces is a single large sphere. Hence, the key to proving theorem 2 is understanding how the assumption of nonnegative scalar curvature on (M^3, g) bounds the behavior of $F(V)$.

Theorem 4 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, is Schwarzschild at infinity, and satisfies condition 2. Then the function $F(V)$ defined in definition 2 satisfies*

$$F''(V) \leq \frac{36\pi - F'(V)^2}{6F(V)}$$

in the sense of comparison functions, where this means that for all $V_0 \geq 0$ there exists a smooth function $F_{V_0}(V) \geq F(V)$ with $F_{V_0}(V_0) = F(V_0)$ satisfying

$$F''_{V_0}(V_0) \leq \frac{36\pi - F'_{V_0}(V_0)^2}{6F_{V_0}(V_0)}$$

Sketch of proof. The method of the proof here is exactly as in theorem 3. The reason condition 2 is needed is that if two components of $\{\Sigma_i(V_0)\}$ push up against each other, then we can not flow both of the surfaces out at the same time. We want to find a flow on $\{\Sigma_i(V_0)\}$ which is constant (but different) on each component. Let $F_{V_0}(V)$ be the sum of the areas to the three halves power of these new surfaces, parameterized as before by the total enclosed volume V. First we consider a flow which is constant on $\Sigma_i(V_0)$ and zero on all the other components. As in inequality 2.3, we get that

$$F''_{V_0}(V_0) \leq \frac{36\pi - F'_{V_0}(V_0)^2}{6A_i(V_0)^{\frac{3}{2}}}$$

The next observation to make is that the value we get for $F'_{V_0}(V_0)$ is independent of which component we flow out. Otherwise, we could find a volume preserving flow which flowed out on one component and flowed in on the other component which decreased the value of F. It follows that $F'_{V_0}(V_0)$ is the same for any flow. From these observations, it is possible to calculate $F''_{V_0}(V_0)$ for any flow which is constant on each component. We then choose the flow which gives us the best estimate for $F''_{V_0}(V_0)$ which is

$$F''_{V_0}(V) \leq \frac{36\pi - F'_{V_0}(V_0)^2}{6\sum_i A_i(V_0)^{\frac{3}{2}}} = \frac{36\pi - F'_{V_0}(V_0)^2}{6F_{V_0}(V_0)}$$

Finally, since $F(V_0) = F_{V_0}(V_0)$ and $F(V) \leq F_{V_0}(V)$ for every $V_0 \geq 0$, the theorem follows as before. □

2.2 The Mass Function

The function $\tilde{F}(V) = A(V)^{\frac{3}{2}}$ will be used to prove theorem 1 and the function $F(V)$ will be used to prove theorem 2. We choose to abuse notation slightly from this point on and call both functions $F(V)$ since both functions satisfy

$$F''(V) \leq \frac{36\pi - F'(V)^2}{6F(V)} \tag{2.4}$$

in the sense of comparison functions. It always will be clear from the context which function is intended. Given an inequality like the one above, it is natural to want to integrate it.

Definition 3 *For $V \geq 0$, let*

$$m(V) = F(V)^{\frac{1}{3}}\left(36\pi - F'(V)^2\right)/c$$

be the mass function, where $c = 144\pi^{\frac{3}{2}}$.

$F(V)$ is continuous, but $F'(V)$ does not necessarily exist for all V, although it does exist almost everywhere since $F(V)$ is monotone increasing. The left and right hand derivatives, $F'_+(V)$ and $F'_-(V)$, do always exist though. This follows from the fact that $F(V)$ has a comparison function $F_{V_0}(V)$ (or $A_{V_0}(V)^{\frac{3}{2}}$) which touches F at

$V = V_0$ and is greater than F in some neighborhood of V_0 for all $V_0 \geq 0$. Since the second derivatives of the comparison functions are uniformly bounded from above in a bounded interval we can add a quadratic to $F(V)$ to get a concave function, from which it follows that the left and right hand derivatives always exist and are equal except at a countable number of points.

Furthermore, $F'_+(V) \leq F'_-(V)$ using the comparison function argument again since $F'_+(V_0) \leq F'_{V_0}(V_0) \leq F'_-(V_0)$. If $F'(V)$ does not exist, then it is natural to define $F'(V)$ to be a multivalued function taking on every value in the interval $(F'_+(V), F'_-(V))$. This is consistent, since if $F'(V)$ does exist, then $F'_+(V) = F'_-(V)$. Hence, $m(V)$ is multivalued for some V, which can be interpreted as the mass "jumping up" at these V, and the set of V for which $m(V)$ and $F(V)$ are multivalued is a countable set. Alternatively, one could replace $F'(V)$ with $F'_+(V)$ (or $F'_-(V)$) in the formula for $m(V)$ so that $m(V)$ would always be single valued.

Lemma 1 *The quantity $m(V)$ is a nondecreasing function of V.*

Proof. The main idea is that if $F(V)$ were smooth,

$$m'(V) = 2F^{\frac{1}{3}}F'(V)\left(-F''(V) + \frac{36\pi - F'(V)^2}{6F(V)}\right)/c$$

being nonnegative would follow from inequality 2.4 and the fact that $F(V)$ is nondecreasing.

More generally, it is sufficient to prove that $m'(V) \geq 0$ distributionally. Hence, treating $m(V)$ as a distribution we may equivalently define

$$m(V) = F(V)^{1/3}\left(36\pi - F'_+(V)^2\right)$$

since $F'_+(V) = F'(V)$ except at a countable number of points. It is convenient to extend $F(V)$ and $m(V)$ to be defined for all real V, so define $F(V) = F(0)$ for $V < 0$. Then since $F'(0) = 0$ we still have

$$F''(V) \leq \frac{36\pi - F'(V)^2}{6F(V)}$$

in the sense of comparison functions for all $V \in (-\infty, \infty)$, where we recall that this means that for all V_0 there exists a smooth function $F_{V_0}(V) \geq F(V)$ with $F_{V_0}(V_0) = F(V_0)$ satisfying

$$F''_{V_0}(V) \leq \frac{36\pi - F'_{V_0}(V_0)^2}{6F_{V_0}(V_0)}. \tag{2.5}$$

To prove that $m'(V) \geq 0$ distributionally, we will prove that

$$-\int_{-\infty}^{\infty} m(V)\phi'(V)\, dV \geq 0$$

for all smooth positive test functions ϕ with compact support. We will need the finite difference operator Δ_δ which we will define as

$$\Delta_\delta(g(V)) = \frac{1}{\delta}(g(V+\delta) - g(V)).$$

Then

$$\begin{aligned}
-\int_{-\infty}^{\infty} m(V)\phi'(V)\, dV &= -\int_{-\infty}^{\infty} F(V)^{1/3}\left(36\pi - F'_+(V)^2\right)\phi'(V)\, dV \\
&= -\lim_{\delta\to 0}\int_{-\infty}^{\infty} F(V)^{1/3}\left(36\pi - (\Delta_\delta F(V))^2\right)(\Delta_\delta\phi(V))\, dV \\
&= -\lim_{\delta\to 0}\int_{-\infty}^{\infty} \Delta_{-\delta}\left\{F(V)^{1/3}\left(36\pi - (\Delta_\delta F(V))^2\right)\right\}\phi(V)\, dV
\end{aligned}$$

where we have used the integration by parts formula for the finite difference operator, $\int f(x)(\Delta_\delta g(x))\, dx = \int g(x)\Delta_{-\delta} f(x)\, dx$, which follows from a change of variables. Then since $F(V)$ has left-hand derivatives everywhere, in the limit, we have

$$= \lim_{\delta\to 0}\int_{-\infty}^{\infty} F(V)^{1/3}\left\{\Delta_{-\delta}[(\Delta_\delta F(V))^2] + F'_-(V)\frac{36\pi - F'_+(V)^2}{3F(V)}\right\}\phi(V)\, dV.$$

Using the comparison functions at each point, since $F_{V_0}(V_0+\delta) \geq F(V_0+\delta)$, $F_{V_0}(V_0-\delta) \geq F(V_0-\delta)$, $F_{V_0}(V_0) = F(V_0)$, and $F(V)$ and $F_{V_0}(V)$ are increasing, it follows that

$$\Delta_{-\delta}\left[(\Delta_\delta F(V_0))^2\right] \geq \Delta_{-\delta}\left[(\Delta_\delta F_{V_0}(V))^2\right]\Big|_{V=V_0}.$$

Changing the integration variable to V_0, then, we have that

$$
\begin{aligned}
&-\int_{-\infty}^{\infty} m(V)\phi'(V)\,dV \geq \\
&\lim_{\delta\to 0}\int_{-\infty}^{\infty} F^{1/3}(V_0)\left\{\Delta_{-\delta}\left[(\Delta_\delta F_{V_0}(V))^2\right]\Big|_{V=V_0} + F'_-(V_0)\frac{36-F'_+(V_0)^2}{3F(V_0)}\right\}\phi(V_0)\,dV_0
\end{aligned}
$$

and since $F'_+(V_0) = F'_-(V_0) = F'_{V_0}(V_0)$ except at a countable number of points,

$$
\begin{aligned}
&= \lim_{\delta\to 0}\int_{-\infty}^{\infty} F^{1/3}(V_0)\left\{\Delta_{-\delta}\left[(\Delta_\delta F_{V_0}(V))^2\right]\Big|_{V=V_0} + F'_{V_0}(V_0)\frac{36-F'_{V_0}(V_0)^2}{3F_{V_0}(V_0)}\right\}\phi(V_0)\,dV_0 \\
&= \int_{-\infty}^{\infty} F^{1/3}(V_0)\left\{-2F'_{V_0}(V_0)F''_{V_0}(V_0) + F'_{V_0}(V_0)\frac{36-F'_{V_0}(V_0)^2}{3F_{V_0}(V_0)}\right\}\phi(V_0)\,dV_0 \\
&= \int_{-\infty}^{\infty} 2F(V_0)^{1/3}F'_{V_0}(V_0)\left\{-F''_{V_0}(V_0) + \frac{36-F'_{V_0}(V_0)^2}{6F_{V_0}(V_0)}\right\}\phi(V_0)\,dV_0 \\
&\geq 0
\end{aligned}
$$

since $F'_{V_0}(V_0) \geq 0$ and the comparison functions satisfy inequality 2.5. Hence, $m'(V) \geq 0$ distributionally, so $m(V)$ is a nondecreasing function of V.$\square$

2.3 Proof of the Penrose Inequality Assuming Condition 1 or 2

The key to the proofs of theorems 1 and 2 is the mass function $m(V)$. First, we consider the context of theorem 1 so that we have only one horizon and we are minimizing area with a volume constraint.

In section 2.6 we show that if M^3 is Schwarzschild with mass m at infinity, then for large V there is a unique area minimizer, and that this minimizer is one of the spherically symmetric spheres of the Schwarzschild metric. Hence, for large V, the functions $F(V) = A(V)^{\frac{3}{2}}$ and hence $m(V)$ are computable in terms of the parameter m, and in fact $m(V) = m$. Also, since the horizon has zero mean curvature, $F'(0) = 0$, so $m(0) = \sqrt{\frac{A}{16\pi}}$, the mass of the black hole.

Since $m(0)$ equals the mass of the black hole and $m(\infty)$ equals the total mass of the system, we now see why it is reasonable to call the function $m(V)$ mass. (In fact, $m(V)$ is equal to the Hawking mass of $\Sigma(V)$, studied by Christodoulou and Yau in [6] and by Huisken and Yau in [16].) Since $m(V)$ is increasing, $m(\infty) \geq m(0)$, so

$$m \geq \sqrt{\frac{A}{16\pi}}$$

and we see that theorem 1, the Penrose inequality for manifolds which satisfy condition 1, is true.

The proof of theorem 2 is exactly the same, but we get a stronger result. We are back in the context of multiple horizons, and we are minimizing the quantity F (from definition 2) given a volume constraint and assuming condition 2. Again, in section 2.6 we show that if M^3 is Schwarzschild with mass m at infinity, then for large V there is a unique F minimizer, and that this minimizer is a single spherically symmetric sphere of the Schwarzschild metric. Thus, once again, $m(V) = m$ for sufficiently large V. However, while $F'(0) = 0$ again since the horizons still have zero mean curvature, $F(0) = \sum_i A_i^{\frac{3}{2}}$, where the $\{A_i\}$ are the areas of the horizons. Hence,

$$m(0) = \left(\sum_{i=1}^{n} \left(\frac{A_i}{16\pi} \right)^{\frac{3}{2}} \right)^{\frac{1}{3}},$$

so this time we get

$$m \geq \left(\sum_{i=1}^{n} \left(\frac{A_i}{16\pi} \right)^{\frac{3}{2}} \right)^{\frac{1}{3}}$$

which proves that theorem 2, the Penrose inequality for manifolds which satisfy condition 2, is true.

2.4 Spherical Symmetry at Infinity

Definition 4 *(M^n, g) is said to be asymptotically flat if there is a compact set $K \subset M$ and a diffeomorphism $\Phi : M - K \to \mathbf{R}^n - \{|x| < 1\}$ such that, in the coordinate chart* *defined by* Φ,

$$g = \sum_{i,j} g_{ij}(x) dx^i dx^j$$

where

$$g_{ij}(x) = \delta_{ij} + O(|x|^{-p})$$

$$|x||g_{ij,k}(x)| + |x|^2 |g_{ij,kl(x)}| = O(|x|^{-p})$$

$$|R(g)| = O(|x|^{-q})$$

for some $p > \frac{n-2}{2}$ *and some* $q > n$*, where we have used commas to denote partial derivatives in the coordinate chart, and* $R(g)$ *is the scalar curvature of* (M^n, g)*.*

These assumptions on the asymptotic behavior of (M^n, g) at infinity imply the existence of the limit

$$M_{ADM}(g) = (4\omega_{n-1})^{-1} \lim_{\sigma \to \infty} \int_{S_\sigma} \sum_{i,j} (g_{ij,i}\nu_j - g_{ii,j}\nu_j)\, d\mu$$

where $\omega_{n-1} = Vol(S^{n-1}(1))$, S_σ is the sphere $\{|x| = \sigma\}$, ν is the unit normal to S_σ in Euclidean space, and $d\mu$ is the Euclidean area element of S_σ. The quantity M_{ADM} is called the *total mass* of (M^n, g) (see [1], [2], [25], and [29]).

Theorem 5 *(Schoen-Yau [29]) Let* (M^n, g)*,* $n \geq 3$*, be a complete asymptotically flat* n*-manifold with* $R(g) \geq 0$*. For any* $\epsilon > 0$*, there is a metric* $\bar{g}$ *such that* $(M^n, \bar{g})$ *is asymptotically flat, and outside a compact set* $(M, \bar{g})$ *is conformally flat and has* $R(\bar{g}) = 0$*, and* $M_{ADM}(\bar{g}) < M_{ADM}(g) + \epsilon$*.*

Furthermore, although Schoen and Yau did not originally mention it, their proof of the above theorem also proves a stronger version of the theorem which we will use, namely that the theorem is still true if we require $|M_{ADM}(\bar{g}) - M_{ADM}(g)| < \epsilon$ and $\bar{g}$ and g to be ϵ-quasi isometric. We say that two metrics on M^n are ϵ-quasi isometric if for all $x \in M^n$

$$e^{-\epsilon} < \frac{\bar{g}(v,v)}{g(v,v)} < e^{\epsilon}$$

for all tangent vectors $v \in T_x(M^n)$.

Since (M, g) is conformally flat and scalar flat outside a compact set, we may choose $\mathbf{R}^n - B_{r_0}(0)$ as a coordinate chart for $\bar{g}$ for some $r_0 > 0$, so that

$$\bar{g}_{ij}(x) = u(x)^{\frac{4}{n-2}}\delta_{ij}, \quad \text{for } |x| > r_0.$$

Since

$$R(\bar{g}) = -\frac{4(n-1)}{n-2}u^{-(\frac{n+2}{n-2})}\Delta u$$

where Δ is the Euclidean Laplacian and $R(\bar{g}) = 0$, we see that $\Delta u = 0$ for $|x| > r_0$. Since (M^n, g) is asymptotically flat, $u(x)$ tends to 1 as x goes to infinity.

Thus, expanding $u(x)$ in terms of spherical harmonics of $\mathbf{R}^n$, we find that

$$u(x) = 1 + \frac{M_{ADM}}{(n-1)|x|^{n-2}} + O(\frac{1}{|x|^{n-1}}).$$

Now we define a new metric $\tilde{g}$, ϵ-quasi isometric to $\bar{g}$ which will be spherically symmetric with zero scalar curvature outside a compact set. To do this, choose any $R > r_0$ and $\delta > 0$ and let

$$v(x) = A + \frac{B}{|x|^{n-2}}$$

where A and B are chosen so that

$$A + \frac{B}{R^{n-2}} = \sup_{|x|=R} u(x) + \delta$$

$$A + \frac{B}{(2R)^{n-2}} = \inf_{|x|=2R} u(x) - \delta.$$

Define

$$w(x) = \begin{cases} u(x) & , \quad |x| < R \\ \min(u(x), v(x)) & , \quad R \leq |x| \leq 2R \\ v(x) & , \quad |x| > 2R \end{cases}$$

This function is continuous since $w(x) = u(x)$ for $|x| = R$ and $w(x) = v(x)$ for $|x| = 2R$. Furthermore since u and v are harmonic and the minimum value of two harmonic functions is weakly superharmonic, w is weakly superharmonic.

Now define $\tilde{w}(x) = w * b$ where b is some smooth, spherically symmetric, positive bump function of total integral 1 and compact support in $B_\delta(0)$. Then $\tilde{w}$ is smooth and superharmonic and $\tilde{w}(x) = u(x)$ for $|x| < R-\delta$ and $\tilde{w}(x) = v(x)$ for $|x| > R+\delta$, since u and v are harmonic and hence have the mean value property.

Let $\tilde{g} = \bar{g}$ everywhere except in the region that $\bar{g}$ is conformally flat and scalar flat. In this region parameterized by $\mathbf{R}^n - B_{r_0}(0)$, let

$$\tilde{g}_{ij}(x) = \tilde{w}^{\frac{4}{n-2}}\delta_{ij}, \quad \text{for } |x| > r_0.$$

Since $\tilde{w}$ is superharmonic,

$$R(\tilde{g}) = -\frac{4(n-1)}{n-2}\tilde{w}^{-\left(\frac{n+2}{n-2}\right)}\Delta\tilde{w} \geq 0.$$

Furthermore, if we choose R big enough and δ small enough we can make $\tilde{g}$ ϵ-quasi isometric to g and

$$|M_{ADM}(\tilde{g}) - M_{ADM}(\bar{g})| < \epsilon$$

for any $\epsilon > 0$. The bound on the mass comes from the fact that

$$M_{ADM}(\tilde{g}) = (n-1)AB$$

and choosing R large and δ small gives A close to 1 and B close to $\frac{M_{ADM}(\bar{g})}{n-1}$. Hence we have the following theorem:

Theorem 6 *Let (M^n, g), $n \geq 3$, be a complete asymptotically flat n-manifold with $R(g) \geq 0$. For any $\epsilon > 0$ there is a metric $\tilde{g}$ ϵ-quasi isometric to g such that (M^n, g) has $R(\tilde{g}) \geq 0$, is asymptotically flat, is spherically symmetric with $R(\tilde{g}) = 0$ outside a compact set, and has $|M_{ADM}(\tilde{g}) - M_{ADM}(g)| < \epsilon$.*

The statement of this theorem can be simplified by introducing the following terminology: We define $(\mathbf{R}^n - \{0\}, h)$ to be the Schwarzschild metric of mass m where

$$h_{ij}(x) = \left(1 + \frac{m}{(n-1)|x|^{n-2}}\right)^{\frac{4}{n-2}}\delta_{ij}.$$

This metric is spherically symmetric, asymptotically flat, has zero scalar curvature, and has total mass m.

Definition 5 *We say that* (M^n, g) *is Schwarzschild with mass* m *at infinity if* $(M^n - K, g)$ *is isometric to* $(\mathbf{R}^n - B, h)$ *for some compact set* K *in* M^n *and some ball* B *in* $\mathbf{R}^n$ *centered around the origin.*

With this definition, the statement of the previous theorem can be stated as follows.

Theorem 7 *Let* (M^n, g) *be a complete asymptotically flat n-manifold with* $R(g) \geq 0$ *and with total mass* M. *For any* $\epsilon > 0$, *there exists a metric* $\tilde{g}$ ϵ*-quasi isometric to* g *with* $R(\tilde{g}) \geq 0$, $(M^n, \tilde{g})$ *Schwarzschild with mass* m *at infinity, and* $|m - M| < \epsilon$.

Hence, since the positive mass and Penrose inequalities are closed conditions, we see by theorem 7 that without loss of generality, we may assume in the statements of the positive mass theorem and the Penrose inequality that the manifolds are Schwarzschild at infinity.

2.5 The Isoperimetric Surfaces of the Schwarzschild Manifold

In the previous section we justified the claim that without loss of generality for proving the Penrose inequality, we could assume that (M^3, g) is isometric to the Schwarzschild metric of mass m outside a compact set. By the positive mass theorem, $m \geq 0$, and since the standard, flat $\mathbf{R}^3$ metric is the only metric with $m = 0$, we generally have $m > 0$. In the next two sections, we prove that the isoperimetric surfaces of M^3, the surfaces $\Sigma(V)$ which minimize area given a volume constraint V, are the spherically-symmetric spheres of the Schwarzschild metric when V is large enough.

We recall that the Schwarzschild metric of mass m is given by $(\mathbf{R}^3 - \{0\}, h)$ where $h_{ij} = (1 + \frac{m}{2r})^4 \delta_{ij}$. The metric is spherically symmetric, asymptotically flat, has zero scalar curvature, and has an outermost minimal sphere at $r = m/2$. In fact, the Schwarzschild metric is symmetric under the mapping $r \to \frac{m^2}{4r}$ and so has two asymptotically flat ends.

Theorem 8 *In the Schwarzschild metric of mass $m \geq 0$, $(\mathbf{R}^3 - \{0\}, h)$, described above, the spherically symmetric spheres given by $r =$ constant minimize area among all other surfaces in their homology class containing the same volume.*

Proof. Since there is an infinite amount of volume inside the horizon of the Schwarzschild metric, we first comment that "containing the same volume" is a well defined notion among surfaces in the same homology class. Equivalently, one could define the volume contained by a surface in the horizon's homology class to be the volume contained by the region outside the horizon keeping track of signs if the region is not entirely outside the horizon.

Let Σ be a spherically symmetric sphere $r = c > m/2$ of the Schwarzschild metric $(\mathbf{R}^3 - \{0\}, h)$. We will prove that Σ is an isoperimetric surface of $(\mathbf{R}^3 - \{0\}, h)$, and the case when $r < m/2$ will follow from the symmetry of the Schwarzschild metric under $r \to \frac{m^2}{4r}$. We omit the case $r = m/2$, but in this case it is easy to show that Σ minimizes area among all surfaces even without the volume constraint (see figure 1.1).

By direct calculation, it is easy to compute that the Hawking mass of Σ is always m, that is

$$\begin{aligned} m &= \left(\frac{A}{16\pi}\right)^{1/2} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2\right) \\ &= \left(\frac{A}{16\pi}\right)^{1/2} \left(1 - \frac{1}{16\pi} H^2 A\right) \end{aligned}$$

where A is the area of Σ and H is the mean curvature of Σ which is constant on Σ by symmetry. Since $m > 0$, $H^2 A < 16\pi$ for Σ, and since $c > m/2$, it is easy to check that $H > 0$. Notice that we have already used the positivity of the mass m.

Now we construct a new metric $(\mathbf{R}^3, k)$ which is isometric to $(\mathbf{R}^3 - \{0\}, h)$ outside Σ but is isometric to a spherically symmetric connected neighborhood of the tip of a spherically symmetric cone inside Σ, where the proportions of the cone are chosen to

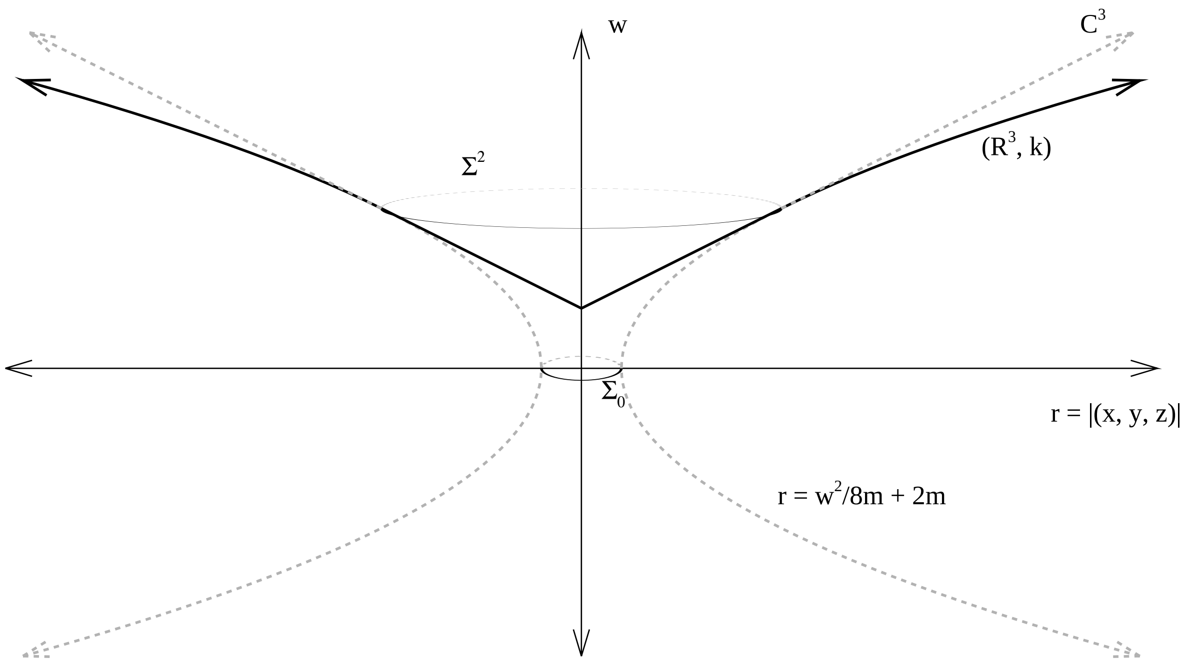

Figure 2.2: Picture of $(\mathbf{R}^3, k)$ isometrically embedded in four-dimensional Euclidean space.

give Σ the same area A and mean curvature H from the inside as the outside. The form of the metric $(\mathbf{R}^3, k)$ can then be written

$$ds_k^2 = u(r)^{-2}dr^2 + u(r)r^2 d\sigma^2$$

in spherical coordinates $(r, \vec{\sigma})$ in $\mathbf{R}^3$, where $d\sigma^2$ is the standard metric on the sphere of radius 1 in $\mathbf{R}^3$. Notice that $u(r) \equiv 1$ would represent the standard flat metric of $\mathbf{R}^3$, and that $u(r) \equiv$ constant gives a cone. But the main point of this form for the metric is that the volume element of ds^2 is the standard volume element in $\mathbf{R}^3$ no matter what $u(r)$ is.

It is helpful to view $(\mathbf{R}^3, k)$ as a submanifold of $\mathbf{R}^4$ (see figure 2.2). We recall from figure 1.1 that the submanifold $|(x, y, z)| = \frac{w^2}{8m} + 2m$ is the Schwarzschild metric of mass m. Let C^3 be the cone in $\mathbf{R}^4$ which is tangent to the Schwarzschild metric on Σ^2. Then $(\mathbf{R}^3, k)$ is the spherically symmetric solid portions shown in figure 2.2, equal to the union of the Schwarzschild metric outside Σ^2 and the cone C^3 inside Σ^2.

Suppose Σ is at $r = \bar{c}$ in this new metric $(\mathbf{R}^3, k)$. Since $(\mathbf{R}^3, k)$ is a cone inside Σ, $u(r) = a$ for $r < \bar{c}$ for some constant a. Since $H^2 A < 16\pi$ for Σ, it turns out that $0 < a < 1$.

In the Schwarzschild metric, if we parameterized the areas of the spherically symmetric spheres by the enclosed volume (outside the horizon) we get from the vanishing of the scalar curvature that $A(V)$ satisfies

$$A^2 A''(V) = 4\pi - \frac{3}{4} A'(V)^2 A \tag{2.6}$$

We will now reparameterize $u(r) = \bar{u}(V)$ where $V = \frac{4}{3}\pi r^3$ is the enclosed volume. This is convenient since the coordinate chart is volume preserving. Hence, since $(\mathbf{R}^3, k)$ is Schwarzschild outside Σ, we have

$$\bar{u}(V) \quad = \quad \begin{cases} a & , \quad V < V_0 \\ \frac{A(V)}{(36\pi)^{1/3} V^{2/3}} & , \quad V \geq V_0 \end{cases}$$

where $V_0 = \frac{4}{3}\pi \bar{c}^3$ and $A(V)$ satisfies the second order differential equation (2.6) with initial conditions $A(V_0)$ and $A'(V_0)$ such that

$$\bar{u}(V_0) = a$$

and

$$\bar{u}'(V_0) = 0.$$

These last two initial conditions guarantee that Σ (which is the sphere at the boundary between the cone and the Schwarzschild metric) has the same area and mean curvatures on the inside and the outside, and that consequently the curvature of $(\mathbf{R}^3, k)$ is bounded on Σ. Finally,

$$\bar{u}(V) = \frac{A(V)}{(36\pi)^{1/3} V^{2/3}}$$

for $V \geq V_0$ since this factor guarantees that the sphere containing a volume V in $(\mathbf{R}^3, k)$ will have area $A(V)$, and hence $(\mathbf{R}^3, k)$ will be Schwarzschild outside Σ.

Lemma 2 *The following inequality holds.*

$$a \leq \bar{u}(V) \leq 1.$$

Proof. First we show that $\bar{u}(V) < 1$. To do this we note that $(\mathbf{R}^3, k)$ has scalar curvature $R(k) \geq 0$ everywhere since $a < 1$. If we redefine $A(V)$ to be the area of the spherically symmetric sphere in $(\mathbf{R}^3, k)$ containing a volume V, then by direct calculation we have

$$A^2 A''(V) \leq 4\pi - \frac{3}{4} A'(V)^2 A.$$

for all $V \geq 0$, with equality outside Σ. Integrating this inequality implies that $m'(V) \geq 0$ where

$$m(V) = \left(\frac{A}{16\pi}\right)^{1/2} \left(1 - \frac{1}{16\pi} A(V) A'(V)^2\right)$$

and hence that $m(V) \geq 0$ since $m(0) = 0$. Thus, $A'(V)^2 < \frac{16\pi}{A}$, from which it follows that

$$A(V)^3 \leq 36\pi V^2$$

which implies $\bar{u}(V) \leq 1$.

The fact that $\bar{u}(V) \geq a$ follows from the fact that $\bar{u}'(V) \geq 0$. We show that $\bar{u}(V)$ is increasing for $V \geq V_0$ by proving that $\bar{u}'(V) \leq 0$ would imply $\bar{u}''(V) \geq 0$. Hence, since $\bar{u}'(V_0) = 0$, it follows that the minimum value of $\bar{u}'(V)$ for $V \geq V_0$ is zero, so $\bar{u}'(V) \geq 0$. We compute for $V \geq V_0$.

$$\bar{u}(V) = \frac{A(V)}{(36\pi)^{1/3} V^{2/3}}$$

$$(36\pi)^{1/3} \bar{u}'(V) = (A'(V) - \frac{2}{3} A V^{-1}) V^{-2/3}$$

so $\bar{u}'(V) \leq 0$ implies

$$A'(V) \leq \frac{2}{3} A V^{-1}$$

But

$$(V^{8/3}A^2(36\pi)^{1/3})\bar{u}''(V) = A^2A''(V)V^2 - \frac{4}{3}A'(V)A^2V + \frac{10}{9}A^3$$

Since $A^2A''(V) = 4\pi - \frac{3}{4}A'(V)^2A$ in the Schwarzschild metrics, it follows that for $V \geq V_0$,

$$(V^{8/3}A^2(36\pi)^{1/3})\bar{u}''(V) = (4\pi - \frac{3}{4}A'(V)^2A)V^2 - \frac{4}{3}A'(V)A^2V + \frac{10}{9}A^3.$$

If $\bar{u}'(V) \leq 0$, then $A'(V) \leq \frac{2}{3}AV^{-1}$, so

$$(V^{8/3}A^2(36\pi)^{1/3})\bar{u}''(V) \geq \frac{1}{9}(36\pi V^2 - A^3) \geq 0$$

from before. Thus, for $V \geq V_0$, $\bar{u}'(V) \geq 0$, so $\bar{u} \geq a$. This completes the proof that $a \leq \bar{u}(V) < 1$, so it follows that $a \leq u(r) \leq 1$ for all r, too. □

Now we prove that Σ at $r = \bar{c}$ is an isoperimetric sphere of $(\mathbf{R}^3, k)$. Let $\tilde{\Sigma}$ be any other surface in $(\mathbf{R}^3, k)$ containing the same volume V_0 as Σ (or greater volume). Let A and $\tilde{A}$ be the $(\mathbf{R}^3, k)$ areas of Σ and $\tilde{\Sigma}$ respectively and let A_0 and $\tilde{A}_0$ be the areas of Σ and $\tilde{\Sigma}$ in the $\mathbf{R}^3$ coordinate chart where $(\mathbf{R}^3, k)$ is represented by

$$ds_k^2 = u(r)^{-2}dr^2 + u(r)r^2d\sigma^2, \quad a \leq u(r) \leq 1.$$

Since this coordinate chart is volume preserving, Σ and $\tilde{\Sigma}$ both contain the same volume V_0 in the $\mathbf{R}^3$ coordinate chart. Hence, by the isoperimetric inequality, $\tilde{A}_0 \geq A_0$. Thus, since $u(r) \geq a$ and $u(r)^{-2} \geq a$, and $u(\bar{c}) = a$, we have

$$\tilde{A} \geq a\tilde{A}_0 \geq aA_0 = A.$$

Hence Σ minimizes area among all surfaces which contain a volume V_0 in $(\mathbf{R}^3, k)$.

Since $(\mathbf{R}^3-\{0\}, k)$ and $(\mathbf{R}^3-\{0\}, h)$, the Schwarzschild metric, are both spherically symmetric, they are conformally equivalent. In fact we can represent $(\mathbf{R}^3 - \{0\}, h)$ as

$$ds_h^2 = w(r)^4\left(u(r)^{-2}dr^2 + u(r)r^2d\sigma^2\right)$$

where $w(r) \equiv 1$ for $r \geq \bar{c}$. Furthermore, by the scalar curvature formula for conformal metrics [25], $\Delta w \geq 0$, so $w(r) > 1$ for $r < \bar{c}$ since Schwarzschild has zero scalar curvature and the cones with $u(r) = a < 1$ have positive scalar curvature.

Now we prove Σ at $r = \bar{c}$ minimizes area among all surfaces in the Schwarzschild metric in its homology class containing the same (relative) volume. Let $\tilde{\Sigma}$ be any other such sphere. Since $\tilde{\Sigma}$ contains the same volume in $(\mathbf{R}^3 - \{0\}, h)$ as Σ, it must contain more volume than Σ in the $(\mathbf{R}^3, k)$ metric under the conformal identification since $w \geq 1$. Hence it has more area in the $(\mathbf{R}^3, k)$ metric since Σ is an isoperimetric sphere of $(\mathbf{R}^3, k)$. Then again, since $w \geq 1$ but $w(\bar{c}) = 1$, $\tilde{\Sigma}$ must have more area than Σ in the Schwarzschild metric $(\mathbf{R}^3 - \{0\}, h)$. Thus, we have proved theorem 8. $\Box$

Now we consider minimizing F, "the sum of the areas to the three halves power" from definition 2, with a volume constraint in the Schwarzschild metric. Using the same argument as in the proof of theorem 8, we find that the collection of surfaces $\{\Sigma_i(V)\}$ which minimizes F among collections of surfaces containing the horizon and a volume V outside the horizon is always a single spherically symmetric sphere of the Schwarzschild metric.

The only real difference in the proof is understanding minimizers of F in $\mathbf{R}^3$. Whereas a sphere minimizes area given a volume constraint in $\mathbf{R}^3$, any collection of spheres minimizes F in $\mathbf{R}^3$ given a volume constraint. This follows from the isoperimetric inequality, $A_i^{\frac{3}{2}} \geq \sqrt{36\pi} V_i$ with equality for spheres. Hence, $\sum_i A_i^{\frac{3}{2}} \geq \sqrt{36\pi} V$ with equality for collections of spheres. However, a single sphere containing a volume V still minimizes F, and so the proof from theorem 8 still applies.

Theorem 9 *In the Schwarzschild metric of mass $m \geq 0$, $(\mathbf{R}^3 - \{0\}, h)$, the spherically symmetric spheres minimize F among all other surfaces in their homology class containing the same volume.*

2.6 Mass and Isoperimetric Spheres at Infinity

Now we consider manifolds (M^3, g) which are Schwarzschild of mass m at infinity (see definition 5), are complete, and have nonnegative scalar curvature. By the positive mass theorem, $m \geq 0$. Since (M^3, g) is isometric to the Schwarzschild metric outside a compact set, we expect that when we minimize area with a volume constraint V, the minimizers are still the spherically symmetric spheres of the Schwarzschild metric when V is large enough. In fact this is the case not only for area minimization, but is also true when we minimize F with a volume constraint. To prove this, we begin with three definitions and a lemma.

Definition 6 *Suppose $\Sigma^2 = \partial U^3 \subset M^3$ minimizes area among all surfaces bounding a compact region of the same volume, $|U^3|$. Then we call Σ^2 an isoperimetric surface of M^3.*

Definition 7 *Suppose $\Sigma^2 = \partial U^3 \subset M^3$ minimizes area among all surfaces bounding a compact region of volume greater than or equal to $|U^3|$. Then we call Σ^2 an outer isoperimetric surface of M^3.*

Definition 8 *A mapping $\phi : A^3 \to B^3$ is area nonincreasing if and only if for all surfaces with boundary $\Sigma^2 \subset A^3$, the area of $\phi(\Sigma^2)$ is less than or equal to the area of Σ^2.*

Lemma 3 *Suppose $\Sigma^2 = \partial U^3$ is a smooth surface in M^3, U^3 is compact, and there exists a C^1, onto, area nonincreasing mapping $\phi : M^3 \to N^3$, which is an isometry outside of the interior of U^3, such that $\phi(\Sigma^2) = \partial(\phi(U^3))$ is an outer isoperimetric surface of N^3. Then Σ^2 is an outer isoperimetric surface of M^3.*

Proof. First we claim that ϕ is volume nonincreasing inside Σ^2. Let $\{e_i\}$ be an orthonormal basis at some point $p \in U^3$, the region contained by Σ^2. Let $G_{ij} = \langle D\phi(e_i), D\phi(e_j)\rangle_{N^3}$, and $\bar{G}_{ij} = G_{ij}^{-1} \cdot \det(G)$. Then for $\vec{v} \in T_p(M^3)$, $v_i^t G_{ij} v_j$ is the square of the length of $D\phi(\vec{v})$ and $v_i^t \bar{G}_{ij} v_j$ is the factor by which areas (more generally, $(n-1)$-volumes) get increased in the direction orthogonal to the unit vector $\vec{v}$. Since we are given that areas are not increased, all the eigenvalues of $\bar{G}$ (which is symmetric and so has all real eigenvalues) are less than or equal to one. Thus $\det \bar{G} \leq 1$, which implies $\det(G) \leq 1$ from the formula for $\bar{G}$. But $\det(G)$ is the factor by which volumes are changed at p, so ϕ is volume nonincreasing inside Σ^2.

Now we prove Σ^2 is outer isoperimetric. Let $\bar{\Sigma}^2$ be any competitor for Σ^2, that is, suppose $\bar{\Sigma}^2$ contains at least as much volume in M^3 as Σ^2. Then upon reflection we see that $\phi(\bar{\Sigma}^2)$ must contain at least as much volume as $\phi(\Sigma^2)$ in M^3 since ϕ is volume nonincreasing. Hence, since $\phi(\Sigma^2)$ is outer isoperimetric, $\text{Area}(\phi(\bar{\Sigma}^2)) \geq \text{Area}(\phi(\Sigma^2))$. But ϕ is an isometry on Σ^2 and area nonincreasing everywhere, so $\text{Area}(\bar{\Sigma}^2) \geq \text{Area}(\Sigma^2)$. □

Using this lemma, we can prove theorem 10

Theorem 10 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, and is Schwarzschild at infinity. Then there exists a V_0 such that for all $V \geq V_0$, the spherically symmetric spheres of the Schwarzschild metric minimize area among all other surfaces in their homology class containing the same volume V (outside the horizons, if any exist).*

Proof. Since M^3 is isometric to the Schwarzschild manifold of mass m outside a compact set, then for some $A_{min} \geq 16\pi m^2$, $\Sigma^2(A) \subset M^3$ exists for $A \geq A_{min}$ and is the spherically symmetric sphere of area A of the Schwarzschild portion of M^3. We claim that the spheres of area

$$A \geq \frac{1}{\pi}\left(\frac{A_{min}}{m}\right)^2 \tag{2.7}$$

must be outer isoperimetric spheres of M^3. Let $\Sigma^2 \subset M^3$ be one of these spheres with area satisfying inequality 2.7. As before in the proof of theorem 8, we construct a spherically symmetric manifold $(\mathbf{R}^3, k)$ which is isometric to the Schwarzschild manifold of mass m outside a spherically symmetric sphere $\bar{\Sigma}^2$ and is isometric to a cone inside $\bar{\Sigma}^2$, where the proportions of the cone are chosen so that $\bar{\Sigma}^2$ has the same area and mean curvature from the inside as the outside. As we proved in theorem 8, $\bar{\Sigma}^2$ is outer isoperimetric in $(\mathbf{R}^3, k)$. We perform this construction so that $(\mathbf{R}^3, k)$ has the same mass as M^3 and $\bar{\Sigma}^2$ has the same area as Σ^2. Hence $(\mathbf{R}^3, k)$ outside $\bar{\Sigma}^2$ is isometric to M^3 outside Σ^2.

We want to construct a map $\phi : M^3 \to (\mathbf{R}^3, k)$ which satisfies the conditions of lemma 3. Define ϕ to be the identity isometry map outside Σ^2 so that $\bar{\Sigma}^2 = \phi(\Sigma^2)$.

Inside Σ^2 we will make ϕ spherically symmetric where M^3 is spherically symmetric, so in the region where ϕ is injective we can characterize ϕ with the function $A(\bar{A})$, where A is the area of the spherically symmetric pre-image in M^3 of the spherically symmetric sphere in $(\mathbf{R}^3, k)$ of area $\bar{A}$. Hence, if we let A_0 be the area of Σ^2 and of $\bar{\Sigma}^2$, then $A(A_0) = A_0$. We define $A(\bar{A})$ for $\bar{A} < A_0$ so that $A'(\bar{A})$ is as small as possible for each $\bar{A} < A_0$ such that ϕ is area nonincreasing.

In fact, since $A \geq \bar{A}$, lengths in the spherically symmetric directions get decreased by a factor of $\left(\frac{\bar{A}}{A}\right)^{1/2}$ by ϕ, so that if we define ϕ to increase lengths by a factor of $\left(\frac{\bar{A}}{A}\right)^{-1/2}$ in the radial direction, ϕ will be area nonincreasing. Hence, volumes will be decreased by ϕ locally by a factor of $\left(\frac{\bar{A}}{A}\right)^{1/2}\left(\frac{\bar{A}}{A}\right)^{1/2}\left(\frac{\bar{A}}{A}\right)^{-1/2} = \left(\frac{\bar{A}}{A}\right)^{1/2}$.

It is convenient to consider the spherically symmetric functions V on M^3 and $\bar{V}$ on $(\mathbf{R}^3, k)$, where V and $\bar{V}$ are volumes enclosed by the corresponding spherically symmetric spheres. We note that $\bar{V}$ is defined everywhere on $(\mathbf{R}^3, k)$ but V is defined only where M^3 is Schwarzschild and hence spherically symmetric. Then it is easy to compute that on the cone, $\bar{A}(\bar{V}) = a(36\pi)^{1/3}\bar{V}^{2/3}$, so that for some constant a

$$\bar{A}'(\bar{V}) = a^{3/2} \cdot \frac{2}{3} \cdot (36\pi)^{1/2} \bar{A}^{-1/2}. \tag{2.8}$$

Furthermore, in the Schwarzschild portion of M^3, since from the Hawking mass we have $m = \left(\frac{A}{16\pi}\right)^{1/2}\left(1 - \frac{A}{16\pi}A'(V)^2\right)$,

$$A'(V) = \sqrt{\frac{16\pi}{A}\left(1 - m\left(\frac{16\pi}{A}\right)^{1/2}\right)} \tag{2.9}$$

Also, since we already noted that ϕ decreases volumes locally by a factor of $\left(\frac{\bar{A}}{A}\right)^{1/2}$,

$$\frac{dV}{d\bar{V}} = \left(\frac{A}{\bar{A}}\right)^{1/2}. \tag{2.10}$$

Then, since

$$\frac{dA}{dV} \cdot \frac{dV}{d\bar{V}} = \frac{dA}{d\bar{A}} \cdot \frac{d\bar{A}}{d\bar{V}},$$

we find that

$$A'(\bar{A}) = a^{-3/2}\sqrt{1 - m\left(\frac{16\pi}{A}\right)^{1/2}} \tag{2.11}$$

for $\bar{A} \le A_0$ with initial condition $A(A_0) = A_0$. Hence, this differential equation determines $A(\bar{A})$.

We claim that $A(0) \ge A_{min}$ if $A_0 \ge \frac{1}{\pi}\left(\frac{A_{min}}{m}\right)^2$ as in inequality (2.7). This will prove that an area nonincreasing map ϕ exists from M^3 to $(\mathbf{R}^3, k)$, where ϕ will be defined to map everything in M^3 inside the spherically symmetric sphere of area $A(0)$ to the tip of the cone $(\mathbf{R}^3, k)$. Actually, this mapping is not C^1 as required in the lemma, but the mapping can be perturbed to be C^1 and still stay area nonincreasing.

We need the inequality

$$\sqrt{b - x} \le \sqrt{b}(1 - \frac{1}{2}x)$$

for $0 \le x \le b \le 1$. Hence,

$$\begin{aligned} A'(\bar{A}) &= a^{-3/2}\sqrt{1 - m\left(\frac{16\pi}{A_0}\right)^{1/2} - \left[m(16\pi)^{1/2}\left(A^{-1/2} - A_0^{-1/2}\right)\right]} \\ &\le a^{-3/2}\sqrt{1 - m\left(\frac{16\pi}{A_0}\right)^{1/2}}\left[1 - \frac{m}{2}(16\pi)^{1/2}\left(A^{-1/2} - A_0^{-1/2}\right)\right] \end{aligned}$$

where we will verify later that $m\left(\frac{16\pi}{A}\right)^{1/2} \le 1$ for all $\bar{A} \ge 0$. Since the dimensions of the cone (including a) were chosen so that $\bar{\Sigma}^2$ had the same mean curvature in $(\mathbf{R}^3, k)$ on the inside as on the outside, $A'(A_0) = 1$ since to first order $A(\bar{A}) \cong \bar{A}$ for $\bar{A}$ near A_0. Thus, a must satisfy

$$1 = A'(A_0) = a^{-3/2}\sqrt{1 - m\left(\frac{16\pi}{A_0}\right)^{1/2}},$$

so

$$A'(\bar{A}) \le 1 - \frac{m}{2}(16\pi)^{1/2}\left(A^{-1/2} - A_0^{-1/2}\right).$$

Hence, if we let $D(\bar{A}) = A(\bar{A}) - \bar{A}$, then

$$D'(\bar{A}) \leq -\frac{1}{2}m(16\pi)^{1/2}\left((D(\bar{A})+\bar{A})^{-1/2} - A_0^{-1/2}\right).$$

Since $D(A_0) = 0$, it follows that $D(\bar{A}) \geq 0$ for $\bar{A} \leq A_0$ and that $D'(\bar{A}) \leq 0$. Thus, $D(\bar{A})$ attains its maximum value at zero. Hence

$$D'(\bar{A}) \leq -\frac{1}{2}m(16\pi)^{1/2}\left((D(0)+\bar{A})^{-1/2} - A_0^{-1/2}\right)$$

so that if we integrate both sides from $\bar{A} = 0$ to $\bar{A} = A_0$ we get

$$\begin{aligned} D(A_0) - D(0) &\leq \int_0^{A_0} -\frac{1}{2}m(16\pi)^{1/2}\left((D(0)+\bar{A})^{-1/2} - A_0^{-1/2}\right) d\bar{A} \\ &= -\frac{1}{2}m(16\pi)^{1/2}\left[2(D(0)+\bar{A})^{1/2} - A_0^{-1/2}\bar{A}\right]_{\bar{A}=0}^{\bar{A}=A_0} \\ &= -\frac{1}{2}m(16\pi)^{1/2}\left[2(D(0)+A_0)^{1/2} - 2D(0)^{1/2} - A_0^{1/2}\right] \end{aligned}$$

so that since $D(A_0) = 0$ and $(D(0)+A_0)^{1/2} \geq A_0^{1/2}$,

$$D(0) + m(16\pi)^{1/2}D(0)^{1/2} \geq \frac{1}{2}m(16\pi)^{1/2}A_0^{1/2}. \tag{2.12}$$

Since $A_0 \geq \frac{1}{\pi}\left(\frac{A_{min}}{m}\right)^2$ from inequality (2.7) and $A_{min} \geq 16\pi m^2$ since the minimal sphere in the Schwarzschild manifold has area $16\pi m^2$,

$$D(0) + m(16\pi)^{1/2}D(0)^{1/2} \geq 32\pi m^2.$$

Hence, $D(0)^{1/2} \geq m(16\pi)^{1/2}$, since the left side of the above inequality is an increasing function of $D(0)$. Thus, plugging this into inequality (2.12) we get

$$2D(0) \geq \frac{1}{2}m(16\pi)^{1/2}A_0^{1/2}$$

so that from inequality (2.7) we have

$$D(0) \geq A_{min}.$$

But $D(0) = A(0) - 0 = A(0)$, so

$$A(0) \geq A_{min}$$

which means that we have stayed in the spherically symmetric portion of M^3 for $0 \leq \bar{A} \leq A_0$. We notice that the spherically symmetric sphere in M^3 of area $A(0)$ gets mapped to the tip of the cone $(\mathbf{R}^3, k)$, so we might as well define ϕ to send everything inside the sphere of area $A(0)$ in M^3 to the tip of the cone. Certainly this is an area nonincreasing map. Thus, we have defined a mapping $\phi : M^3 \to (\mathbf{R}^3.k)$ which is an isometry outside of Σ in M^3 and which is area decreasing inside of Σ in M^3. The mapping is not C^1 on the sphere of area $A(0)$ in M^3, but ϕ can be perturbed slightly around the sphere of area $A(0)$ so that it is C^1 and still area nonincreasing. Since $\bar{\Sigma}$ is outer isoperimetric in $(\mathbf{R}^3, k)$ and $\phi(\Sigma) = \bar{\Sigma}$, it follows from lemma 3 that Σ is outer isoperimetric in M^3 and hence minimizes area among surfaces containing the same volume in M^3. Since Σ was any of the spherically symmetric spheres of area $A \geq \frac{1}{\pi}\left(\frac{A_{min}}{m}\right)^2$, this proves theorem 10. $\square$

Since spheres are F-minimizers given a volume constraint, we mentioned in the previous section that $\bar{\Sigma}^2$ minimizes F as well as area given a volume constraint in $(\mathbf{R}^3, k)$. Thus, lemma 3 implies theorem 11 as well as theorem 10.

Theorem 11 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, and is Schwarzschild at infinity. Then there exists a V_0 such that for all $V \geq V_0$, the spherically symmetric spheres of the Schwarzschild metric minimize F among all other surfaces in their homology class containing the same volume V (outside the horizons, if any exist).*

Theorems 8, 9, 10, and 11 are also true in higher dimensions except the exponent in the definition of F is more generally $\frac{n}{n-1}$ instead of $\frac{3}{2}$. Also, note that theorems 10 and 11 are not true if we merely require (M^3, g) to be Schwarzschild at infinity and drop the conditions that (M^3, g) is complete and has nonnegative scalar curvature. We need $R(g) \geq 0$ and completeness to use the positive mass theorem to conclude that $m \geq 0$, which is essential for theorems 8 and 9. In fact, for $m < 0$, the spherically symmetric spheres of the Schwarzschild metric are unstable and hence do not minimize area among surfaces enclosing the same volume.

Theorems 10 and 11 are important because they allow us to evaluate $\lim_{V\to\infty} m(V)$. In fact, since the minimizing surface (when minimizing area) or collection of surfaces (when minimizing F) enclosing a volume V outside the horizons is always a spherically symmetric sphere of the Schwarzschild metric for $V \geq V_0$, $m(V) = m$, the mass parameter of the Schwarzschild metric, for $V > V_0$.

Theorem 12 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, and is Schwarzschild with mass m at infinity. Then for both definitions of $m(V)$ in section 3 (whether we are minimizing area or F with a volume constraint), we have*

$$\lim_{V\to\infty} m(V) = m.$$

This theorem is true in higher dimensions as well. We also conjecture that theorem 12 is true for asymptotically flat manifolds with positive total mass, where m is replaced by the total mass M_{ADM}. With additional decay conditions on the asymptotic flatness of (M^3, g), Huisken and Yau show that the region at infinity is foliated by constant mean curvature spheres which are stable and hence locally minimize area with a volume constraint [16]. We conjecture that these same spheres also globally minimize area among surfaces in the same homology class containing the same volume.

2.7 Existence of Surfaces which Minimize Area Given a Volume Constraint

In our proof of theorem 1, we used the fact that the mass function $m(V)$ is nondecreasing which relies on equation 2.3, which in turn followed from doing a unit normal flow on $\Sigma(V)$, the surface which minimizes area among surfaces containing a volume V outside the horizon. Thus, it is essential that the surface $\Sigma(V)$ actually exists.

In this section, we assume the hypotheses of theorem 1 again, including condition

1, and recall the definition of $A(V)$ given in the introduction to this chapter. We will prove that for all $V \geq 0$, there exists a surface $\Sigma(V)$ which encloses a volume V outside the horizon and which minimizes area, so that $\text{Area}(\Sigma(V)) = A(V)$.

Existence theory for compact manifolds is well understood using geometric measure theory, since the space of rectifiable currents of bounded mass on compact manifolds is compact. The main problem with this type of existence question is that (M^3, g) is not compact. However, we will be able to use the mass function $m(V)$ to combat this problem and prove that the minimizers always exist and lie inside a bounded domain for each V.

First we will prove existence of $\Sigma(V)$ for $0 \leq V \leq V_{MAX}$, but the approach will work for all nonnegative V_{MAX}. Now consider $M^3 \cup S^3$, where the union is a disjoint union and S^3 is a constant curvature 3-sphere with total volume $V_S \gg V_{MAX}$. The approach will be to prove existence of an area minimizer on this manifold, $M^3 \cup S^3$, for volumes less than or equal to V_{MAX}, and then to use the mass function $m(V)$ to prove that the minimizers actually contain zero volume in the S^3 if we choose V_S to be large enough.

For the moment, let us redefine $A(V)$ to be exactly as before in definition 1, except that we replace M^3 with $M^3 \cup S^3$ and $\tilde{M}^3$ with $\tilde{M}^3 \cup S^3$. Since $A(V)$ is the infimum of the areas of surfaces which contain a volume V outside the horizon, there exists a sequence of surfaces $\{\Sigma_i\}$ in $M^3 \cup S^3$, each containing a volume V, and whose areas approach $A(V)$ from above. Again, since $M^3 \cup S^3$ is not compact, we can not conclude that the sequence converges to a limit surface with area $A(V)$. However, using the two propositions below, we will be able to modify this sequence of surfaces so that the areas still converge to $A(V)$ and each surface stays inside a compact region. Then since the space of rectifiable currents with bounded mass in a compact region is compact, we will get a limit surface in $M^3 \cup S^3$ with area $A(V)$.

The first proposition uses the fact that the Schwarzschild metric becomes very flat as we move out to infinity. Recall that the Schwarzschild metric of mass m is $(\mathbf{R}^3 - \{0\}, h)$, where $h_{ij} = (1 + \frac{m}{2r})^4 \delta_{ij}$ and r is the radial coordinate in $\mathbf{R}^3$. In the next proposition, we allow m to be positive, zero, or negative. If m is negative, then the Schwarzschild metric has a singularity at $r = -\frac{m}{2}$. If m is positive, then the metric has a horizon at $r = \frac{m}{2}$. In these cases, for the purposes of the two propositions below, we will say that a surface contains a volume V when it contains a volume V outside the horizon or singularity.

Proposition 1 *Consider the Schwarzschild metric $(\mathbf{R}^3 - \{0\}, h)$ of mass m disjoint union a constant curvature 3-sphere with volume V_S. Then there exists an r such that if we choose any $r_1 > r$ and let $r_2 = 2r_1$, then if Σ is any connected surface containing a volume $V \leq V_S$ intersecting both the coordinate sphere of radius r_1 and the coordinate sphere of radius r_2 (using $\mathbf{R}^3$ coordinates here), then we can modify Σ outside of the coordinate ball of radius r_1 to be three surfaces Σ_1, Σ_2, and Σ_3, with Σ_1 and Σ_2 in the closed 3-dimensional region contained by Σ and with Σ_3 in the constant curvature sphere, such that Σ_1 intersects the coordinate sphere of radius r_1 but not the coordinate sphere of radius r_2, Σ_2 intersects the coordinate sphere of radius r_2 but not the coordinate sphere of radius r_1, and $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ has less area than Σ but still contains the same volume V.*

The main idea of this proposition is that if r_1 and r_2 are large enough, then any connected surface with a finite volume V intersecting both spheres must have at least one very long tentacle. Since Σ has finite total volume, these tentacles must get very thin. Then we can snip the tentacles somewhere in the region between the two spheres so that we get two surfaces, Σ_1 and Σ_2, with Σ_1 entirely inside the coordinate ball of radius r_2 and Σ_2 entirely outside the coordinate ball of radius r_1. The simplest snipping process would simply be to remove a section of the tentacle. By doing this, we've decreased the volume by ΔV, so we define Σ_3 to be a constant curvature 2-sphere of volume ΔV in the constant curvature 3-sphere. Thus the total enclosed volume stays the same, and if we snip the tentacle correctly where we remove a sufficiently long and skinny section, the total area will decrease. Proposition 1 follows as a generalization of theorem 17 which is proved in section 2.9. We leave the details to the reader.

Proposition 2 *Consider the Schwarzschild metric $(\mathbf{R}^3 - \{0\}, h)$ of mass $m \geq 0$ disjoint union a constant curvature 3-sphere with volume V_S. Then there exists an $\tilde{r}$*

such that if Σ is any surface bounding a region of volume $V \leq V_S$ entirely outside the coordinate ball of radius $\tilde{r}$, then the area of Σ is greater than the area of a constant curvature 2-sphere containing a volume V in the constant curvature 3-sphere.

This proposition follows from the fact that Schwarzschild is very nearly flat outside a coordinate ball of large radius. Thus we get that surfaces nearly satisfy the isoperimetric inequality for surfaces in $\mathbf{R}^3$, that $A^{\frac{3}{2}} \geq \sqrt{36\pi}V$, whereas constant curvature 2-spheres in any constant curvature 3-sphere always have $A^{\frac{3}{2}} < \sqrt{36\pi}V$. Making this idea rigorous is delicate, particularly for small volumes. We prove proposition 2 later in section 2.9.

Now we are ready to prove existence of $\Sigma(V)$ on $M^3 \cup S^3$. Again, since $A(V)$ is the infimum of the areas of surfaces which contain a volume V outside the horizon, there exists a sequence of surfaces $\{\Sigma_i\}$ in $M^3 \cup S^3$, each containing a volume V, and whose areas approach $A(V)$ from above. Note that if we modify the surfaces in the sequence in a way which preserves their enclosed volumes but decreases their areas, then the areas of the surfaces still approaches $A(V)$.

The first modification we will make to each surface in the sequence is to take whatever volume is in S^3 and to change that part of the surface to be a single constant curvature 2-sphere in the S^3 enclosing that volume. This always decreases area since it is known that 2-spheres minimize area with a volume constraint in S^3. We will repeat this step whenever more volume is sent to S^3 from M^3 in subsequent modifications of the surfaces.

Next we use propositions 1 and 2 to modify each surface in the sequence. By assumption, M^3 is isometric to the Schwarzschild metric outside a compact set. Again, we use the standard coordinate chart for the region of M^3 which is Schwarzschild, just as we did in propositions 1 and 2. Now we choose r_1 to be greater than the r of proposition 1 and the $\tilde{r}$ of proposition 2 and large enough that M^3 is Schwarzschild outside the coordinate sphere of radius r_1. As in proposition 1, $r_2 = 2r_1$. By proposition 1, we can modify each surface in the sequence so that each component of each surface is either entirely inside the coordinate ball of radius r_2 or entirely outside the coordinate ball or radius r_1. A portion of the volume gets sent to S^3, but the total volume of the surfaces stays the same and the total area decreases.

Next, using proposition 2, we take all of the components of the surfaces outside the coordinate ball of radius r_1 and send them to spheres of the same volume in S^3, one at a time. By proposition 2, this also decreases the areas and preserves the volumes of the surfaces in the sequence. We send these components to S^3 one at a time in the sense that if at any point there are two spheres (or any other surface that is not a single constant curvature sphere) in S^3, we immediately turn this portion of the surface into one constant curvature sphere in S^3 with the same volume. This always decreases the area, preserves volume, and guarantees that there will be room for more spheres to be sent to S^3.

But now every surface in the sequence is contained in the coordinate ball of radius r_2 union S^3. Since the sequence of surfaces in now contained in a compact set and the areas of the surfaces still converge to $A(V)$ from above, it follows from the compactness of the space of rectifiable currents in a compact manifold that a limit surface $\Sigma(V)$ exists and that $\text{Area}(\Sigma(V)) = A(V)$.

Theorem 13 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, contains a single outermost minimal sphere Σ_0, is Schwarzschild at infinity, and satisfies condition 1. Let $\tilde{M}^3$ be the closure of the component of $M^3 - \Sigma_0$ that contains the asymptotically flat end, and let S^3 be a constant curvature sphere of volume V_S. Define*

$$A(V) = \inf_{\Sigma} \{ \mathit{Area}(\Sigma) \mid \Sigma \textit{ contains a volume } V \textit{ outside } \Sigma_0 \}$$

where Σ is the boundary of some 3-dimensional region in $M^3 \cup S^3$ and Σ is a surface in $\tilde{M}^3 \cup S^3$ in the same homology class of $\tilde{M}^3 \cup S^3$ as the horizon Σ_0.

Then for all $V \in [0, V_S]$, there exists a surface $\Sigma(V)$ containing a volume V outside Σ_0 in the same class of surfaces just described such that $\mathit{Area}(\Sigma(V)) = A(V)$.

Now we will prove existence of $\Sigma(V)$ on M^3 for $0 \le V \le V_{MAX}$, for any nonnegative V_{MAX}. Again, consider $M^3 \cup S^3$, where the union is a disjoint union and S^3 is a constant curvature 3-sphere with total volume V_S much bigger than V_{MAX}. We will describe how much bigger in a moment. Since we have already proven existence of a minimizer $\Sigma(V)$ on $M^3 \cup S^3$ for volumes up to V_S, we certainly have existence on the same manifold up to the volume V_{MAX}. Furthermore, by condition 1, we can choose $\Sigma(V)$ to have at most two components, with only one component in M^3 and possibly one component in S^3.

If we choose V_S to be large enough, we can use the mass function $m(V)$ to prove that the minimizers actually contain zero volume in the S^3, and hence are entirely contained in M^3. Let $\tilde{V}$ be the supremum of all volumes $\bar{V} \leq V_{MAX}$ with the property that $\Sigma(V)$ has zero volume in the S^3 for $0 \leq V \leq \bar{V}$. Since we are assuming M^3 satisfies condition 1, M^3 has exactly one horizon, and $\Sigma(0)$ is this horizon, which of course is contained entirely in M^3. Hence, $\tilde{V} \geq 0$.

Furthermore, for $0 \leq V \leq \tilde{V}$, $\Sigma(V)$ has zero volume in the S^3 and hence is in M^3 and has only one component. Thus, by lemma 1, $m(V)$ is a nondecreasing function of V in this range, and since $m(0) = \sqrt{\frac{A}{16\pi}}$, where A is the area of the horizon, $m(V)$ is positive for $0 \leq V \leq \tilde{V}$.

For $V \geq \tilde{V}$, $m(V)$ is no longer necessarily nondecreasing. However, if we reexamine the proof of lemma 1 and the derivation of inequality 2.3, it turns out that there exists a uniform $\epsilon > 0$ which is only a function of V_{MAX} and the area of the horizon such that $m(V) \geq \epsilon$ for $0 \leq V \leq \tilde{V} + \epsilon$.

The reason for this is that in this range, $m'(V)$ can be bounded below uniformly in terms of V_{MAX} and the area of the horizon. Inequality 2.2 is changed where the 4π is replaced by an 8π because the minimizers on which we do a unit normal variation in section 2.1 may now have up to two components, so the Euler characteristic may be as large as 4. The function $A(V)$ is bounded on both sides since it is larger than the area of the horizon and smaller than the area of the horizon plus $(36\pi)^{\frac{1}{3}} V_{MAX}^{\frac{2}{3}}$. The upper bound on $A(V)$ comes from comparing $\Sigma(V)$ with a surface which is the horizon union a roughly spherical surface containing a volume V very far out on the asymptotically flat end of M^3. Finally, since the horizon is outermost, $A'(V)$ is bounded below by zero and bounded above for ϵ small enough since we have an upper bound on $A''(V)$ from inequality 2.2. We leave the details of this to the interested reader.

Since the mass function $m(V) \geq \epsilon$, it follows from definition 3 that $F'(V) = \frac{3}{2} A(V)^{\frac{1}{2}} A'(V) \leq \sqrt{36\pi} - \epsilon'$ for some $\epsilon' > 0$, which is equivalent to $A'(V) \leq \sqrt{\frac{16\pi}{A(V)}} - \epsilon''$

for some uniform $\epsilon'' > 0$. On the other hand, consider $\Sigma(V)$ where $0 \le V \le \tilde{V} + \epsilon$. The surface $\Sigma(V)$ has constant mean curvature $H(V)$ on all the components, and by looking at a unit normal variation of $\Sigma(V)$ and comparing the areas of the variation surfaces with $A(V)$, we get that

$$H(V) \le \sqrt{\frac{16\pi}{A(V)}} - \epsilon'' \tag{2.13}$$

where $\epsilon'' > 0$ is a function of V_{MAX} and the area of the horizon only.

The mean curvature H of a constant curvature sphere of area A in $\mathbf{R}^3$ is $\sqrt{\frac{16\pi}{A}}$. Furthermore, the mean curvature H of a constant curvature sphere of area A in a constant curvature 3-sphere S^3 of volume V_S is as close to the $\mathbf{R}^3$ value as we like if we choose V_S to be large enough. Now suppose $\Sigma(V)$, with $0 \le V \le \tilde{V} + \epsilon$, had a component in S^3. This component is a constant curvature sphere, and we can define V_S in terms of V_{MAX} and the area of the horizon to be large enough so that the mean curvature of the sphere cannot satisfy inequality 2.13. Hence, we have a contradiction, so $\Sigma(V)$ is in M^3 for $0 \le V \le \tilde{V} + \epsilon$.

But $\tilde{V}$ is the supremum of all volumes $\bar{V} \le V_{MAX}$ with the property that $\Sigma(V)$ is entirely contained in M^3 for $0 \le V \le \bar{V}$. Hence, since ϵ is a function of V_{MAX} and the area of the horizon only, $\tilde{V} = V_{MAX}$, proving that the minimizer $\Sigma(V)$ exists in M^3 for all $V \ge 0$ since V_{MAX} was arbitrary.

Theorem 14 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, contains a single outermost minimal sphere Σ_0, is Schwarzschild at infinity, and satisfies condition 1. Let $\tilde{M}^3$ be the closure of the component of $M^3 - \Sigma_0$ that contains the asymptotically flat end. Define*

$$A(V) = \inf_{\Sigma}\{Area(\Sigma) \mid \Sigma \text{ contains a volume } V \text{ outside } \Sigma_0\}$$

where Σ is the boundary of some 3-dimensional region in M^3 and Σ is a surface in $\tilde{M}^3$ in the same homology class of $\tilde{M}^3$ as the horizon Σ_0.

Then for all $V \ge 0$ there exists a surface $\Sigma(V)$ containing a volume V outside Σ_0 in the same class of surfaces just described such that $Area(\Sigma(V)) = A(V)$.

2.8 Existence of Surfaces which Minimize F Given a Volume Constraint

We go back to the case that M^3 has any number of horizons and assume the hypotheses of theorem 2, including condition 2. Let $\tilde{M}^3$ be the closure of the component of $M^3 - \{\text{the horizons}\}$ that contains the asymptotically flat end. Let

$$F(V) = \inf_{\{\Sigma_i\}} \{\sum_i \text{Area}(\Sigma_i)^{\frac{3}{2}} \mid \{\Sigma_i\} \text{ contain a volume } V \text{ outside the horizons}\}$$

where the $\{\Sigma_i\}$ are the boundaries of the components of some 3-dimensional open region in M^3 and $\bigcup_i \Sigma_i$ is in $\tilde{M}^3$ and is in the homology class of $\tilde{M}^3$ which contains both a large sphere at infinity and the union of the horizons. If the collection $\{\Sigma_i\}$ contains a volume V outside the horizons and $\sum_i \text{Area}(\Sigma_i)^{\frac{3}{2}} = F(V)$, then we say that $\{\Sigma_i\}$ minimizes F for the volume V.

In this section we will prove that if M^3 satisfies condition 2, then an F-minimizer always exists. Existence of an F-minimizer for all volumes $V \geq 0$ is necessary to prove theorem 2 since the theorem relied on the fact that we had an increasing mass function $m(V)$. The proof that the mass function was increasing though relied on doing a variation of the F-minimizers for each $V \geq 0$. Thus, it is essential that their exists a collection of surfaces $\Phi(V) = \{\Sigma_i(V)\}$ which minimize F among collections of surfaces in the correct homology class containing a volume V outside the horizon.

We will abuse notation slightly again and define

$$F(\Phi) = \sum_i \text{Area}(\Sigma_i)^{\frac{3}{2}}$$

where $\Phi = \{\Sigma_i\}$ is any collection of surfaces in M^3 which are the boundaries of the components of some 3-dimensional open region in M^3.

First we will prove existence of $\Phi(V)$ for $0 \leq V \leq V_{MAX}$, but the approach will work for all nonnegative V_{MAX}. Now consider $M^3 \cup S^3$, where the union is a disjoint union and S^3 is a constant curvature 3-sphere with total volume $V_S \gg V_{MAX}$. The approach will be to prove existence of an F-minimizer on this manifold, $M^3 \cup S^3$, for volumes less than or equal to V_{MAX}, and then to use the mass function $m(V)$ to prove that the minimizers actually contain zero volume in the S^3 if we choose V_S to be large enough.

For the moment, let us redefine $F(V)$ to be exactly as above except that we replace M^3 with $M^3 \cup S^3$ and $\tilde{M}^3$ with $\tilde{M}^3 \cup S^3$. Since $F(V)$ is the infimum of the F-values of collections of surfaces which contain a volume V outside the horizon, there exists a sequence of collections of surfaces $\{\Phi_i\}$ in $M^3 \cup S^3$, each containing a volume V, and whose F-values approach $F(V)$ from above.

In the previous section, we used propositions 1 and 2 to show that we could modify any sequence of surfaces in $M^3 \cup S^3$ to lie inside a compact region of $M^3 \cup S^3$ without increasing the areas of any of the surfaces. The technique, though, did increase the number of the components of the surfaces. However, since the total area went down and the number of components went up, it follows from the fact that $(a+b)^{\frac{3}{2}} > a^{\frac{3}{2}} + b^{\frac{3}{2}}$ for a and b positive that these same techniques can be used to modify a sequence of collections of surfaces $\{\Phi_i\}$ so that the sequence lies inside a compact region of $M^3 \cup S^3$ without increasing the F-values of any of the collections of surfaces. Also, since it can be checked by direct calculation that the collection of surfaces which minimizes F inside S^3 is a single spherically symmetric sphere, each of the new modified collections of surfaces will have at most one component in the S^3 which will always be a spherically symmetric sphere as before. Since the F-values are not increased, the F-values of the new modified sequence $\{\Phi_i\}$ still converge to $F(V)$ from above.

In the introduction to this chapter we commented that there were two problem to look out for in the existence of F-minimizers. The first is that a component of the F-minimizer could run off to infinity. This problem is taken care of since we are able to require our minimizing sequence to stay inside a compact set. The other problem with F-minimization, though, is that "bubbling" might occur, where the optimal configuration is an infinite number of tiny balls with a finite total volume. To combat this, we modify the sequence $\{\Phi_i\}$ one last time. We know bubbling cannot happen in the S^3, since, by direct calculation, the F-minimizers in S^3 are single spherically symmetric spheres. Hence, we modify a given collection of surfaces $\{\Sigma_i\}$ using the following rule which we will call the "sphere replacement rule". If any subcollection of the surfaces in M^3 would have smaller F-value by replacing them with a single sphere in S^3, then we make the replacement. This rule puts an upper bound on how much volume can be used for tiny balls in M^3 since at some point F can be reduced by replacing a large number of tiny balls in M^3 by a single sphere with the same volume in S^3. The reason for this is that since M^3 is smooth, it has bounded curvature and hence on the small scale is approximately flat. Since F scales like volume, it follows that a bunch of tiny balls containing a volume V will have F-value close to $\sqrt{36\pi}V$, which by direct calculation is larger than the F-value of a single sphere in S^3 containing the same volume. And as before, if at any time there are two or more spheres in S^3, then we combine them into one sphere containing the same volume and this also always decreases the F-value.

Now we are ready to take a limit of a subsequence of $\{\Phi_i\}$. This is a little tricky since each Φ_i is not a surface, but a collection of connected surfaces. For each i, order the surfaces of each collection Φ_i by the volume outside the outermost horizons enclosed by each surface, with largest volumes first. If two surfaces enclose the same volume, then choose either ordering. By the Federer-Fleming compactness theorem, there exists a subsequence $\{\Phi_{1,i}\}$ of $\{\Phi_i\}$ in which the largest surfaces of each Φ converge to a limit. Similarly, there exists a subsequence $\{\Phi_{2,i}\}$ of $\{\Phi_{1,i}\}$ in which the second largest surfaces of each Φ converge to a limit. Repeating this process we define the sequence $\{\Phi_{n,i}\}$ for $n \geq 1$. Finally, we define $\{\tilde{\Phi}_i\} = \{\Phi_{i,i}\}$, which has the property that the largest surfaces converge to a limit, the second largest surfaces converge to a limit, and so on, and we define the collection of the limit surfaces to be $\tilde{\Phi}$.

While we do get a collection of limit surfaces $\tilde{\Phi}$, we still need to show that they enclose the correct volume V that each collection of surfaces in the original sequence enclosed. Suppose $\tilde{\Phi}$ did not enclose the volume V but instead only enclosed a volume $V - v$ for some $v > 0$. Then for any $\epsilon > 0$ there must exist an i such that in the collection Φ_i there is a large subcollection of tiny surfaces each containing less than ϵ volume each but containing a total volume of v. In other words, bubbling has occurred. But by the sphere replacement rule, this can not happen, since the F-value of Φ_i would have been reduced by replacing the large subcollection of tiny surfaces by a single sphere in S^3 containing a volume v, for some value of $\epsilon > 0$. Hence, $\tilde{\Phi}$ encloses a volume V, and $F(\tilde{\Phi}) = F(V)$.

Theorem 15 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, contains any number of outermost minimal spheres $\{\tilde{\Sigma}_i\}$, is Schwarzschild at infinity, and satisfies condition 2. Let $\tilde{M}^3$ be the closure of the component of $M^3 - \{\tilde{\Sigma}_i\}$ that contains the asymptotically flat end, and let S^3 be a constant curvature sphere of volume V_S. Define*

$$F(V) = \inf_{\{\Sigma_i\}} \{\sum_i Area(\Sigma_i)^{\frac{3}{2}} \mid \{\Sigma_i\} \text{ contain a volume } V \text{ outside the horizons}\}$$

where the $\{\Sigma_i\}$ are the boundaries of the components of some 3-dimensional open region in $M^3 \cup S^3$ and $\bigcup_i \Sigma_i$ is in $\tilde{M}^3 \cup S^3$ and is in the homology class of $\tilde{M}^3 \cup S^3$ which contains both a large sphere at infinity and the union of the horizons.

Then for all $V \in [0, V_S]$, there exists a collection of surfaces $\Phi(V) = \{\Sigma_i(V)\}$ containing a volume V outside the horizons in the same class of surfaces just described such that $F(\Phi(V)) = F(V)$.

Now we will prove existence of $\Phi(V)$ on M^3 for $0 \le V \le V_{MAX}$, for any nonnegative V_{MAX}. Again, consider $M^3 \cup S^3$, where the union is a disjoint union and S^3 is a constant curvature 3-sphere with total volume V_S much bigger than V_{MAX}. We will describe how much bigger in a moment. Since we have already proven existence of a minimizer $\Phi(V)$ on $M^3 \cup S^3$ for volumes up to V_S, we certainly have existence on the same manifold up to the volume V_{MAX}. Furthermore, by condition 2, we can choose the collection of surfaces $\Phi(V)$ so that no two of its surfaces touch. Hence, we can perform unit normal variations on each surface of the collection, so the mass function $m(V)$ is nondecreasing as long as we require $V_S \ge 2V_{MAX}$ (since we need the mean curvature of the minimizers to be positive to get $F'(V)$ nonnegative which is required for nondecreasing mass).

Since the mass function is initially positive since $m(0) = \left(\sum_{i=1}^n \left(\frac{A_i}{16\pi}\right)^{\frac{3}{2}}\right)^{\frac{1}{3}}$, and since m is nondecreasing, $m(V)$ is always positive for $0 \le V \le V_{MAX}$. Since $m(V) =$

$F(V)^{\frac{1}{3}}(36\pi - F'(V)^2)/c$ where $c = 144\pi^{\frac{3}{2}}$, it follows that $F'(V) < \sqrt{36\pi} - \epsilon$ for $0 \le V \le V_{MAX}$ for some $\epsilon > 0$.

On the other hand, suppose the surfaces $\{\Sigma_i\}$ minimize F while enclosing a volume V. It follows from the first variation of area on each surface that each surface has constant (generally distinct) mean curvature H_i. Furthermore, from this same first variational computation it follows that if we consider any smooth variation on these surfaces, the rate of change of F with respect to V will be

$$\frac{dF}{dV} = \frac{3}{2} A_i^{\frac{1}{2}} H_i$$

for all i. By comparing this variation with other minimizers, it follows that

$$\frac{3}{2} A_i^{\frac{1}{2}} H_i \le \text{the left sided derivative of } F(V) \le \sqrt{36\pi} - \epsilon \tag{2.14}$$

for some fixed $\epsilon > 0$.

But if we choose V_S to be large enough, then the local geometry of the sphere S^3 can be made as close to that of $\mathbf{R}^3$ as we like. Hence, for a sphere containing a volume less than V_{MAX} in S^3, $\frac{3}{2}A^{\frac{1}{2}}H$ can be made as close to $\frac{3}{2}\sqrt{4\pi r^2}\frac{2}{r} = \sqrt{36\pi}$ as we like if we choose V_S large enough, violating inequality 2.14. Hence, if we choose V_S large enough, then the minimizer $\Phi(V) = \{\Sigma_i(V)\}$ will not have any components in the S^3, which proves that $\Phi(V)$ minimizes F in M^3 among all other collections of surfaces in M^3 in the correct homology class containing the same volume V, for $0 \le V \le V_{MAX}$. But since V_{MAX} was arbitrary, we have a F-minimizer for all $V \ge 0$.

Theorem 16 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, contains any number of outermost minimal spheres $\{\tilde{\Sigma}_i\}$, is Schwarzschild at infinity, and satisfies condition 2. Let $\tilde{M}^3$ be the closure of the component of $M^3 - \{\tilde{\Sigma}_i\}$ that contains the asymptotically flat end. Define*

$$F(V) = \inf_{\{\Sigma_i\}} \{\sum_i \mathit{Area}(\Sigma_i)^{\frac{3}{2}} \mid \{\Sigma_i\} \textit{ contain a volume } V \textit{ outside the horizons}\}$$

where the $\{\Sigma_i\}$ are the boundaries of the components of some 3-dimensional open *region in M^3 and $\bigcup_i \Sigma_i$ is in $\tilde{M}^3$ and is in the homology class of $\tilde{M}^3$ which contains both a large sphere at infinity and the union of the horizons.*

Then for all $V \geq 0$, there exists a collection of surfaces $\Phi(V) = \{\Sigma_i(V)\}$ containing a volume V outside the horizons in the same class of surfaces just described such that $F(\Phi(V)) = F(V)$.

2.9 Another Isoperimetric Inequality for the Schwarzschild Metric

In section 2.6, we proved that the spherically symmetric spheres of the Schwarzschild metric minimize area among all surfaces in their homology class containing the same volume outside the horizon. This gives a lower bound for the area of any surface containing the horizon in terms of the volume outside the horizon that the surface encloses, and so is an isoperimetric inequality.

In this section, we lead up to proving proposition 2 of section 2.7 which is an isoperimetric inequality for the asymptotically flat portion of the Schwarzschild metric since it gives lower bounds for the areas of surfaces in terms of their enclosed volumes. We also prove theorem 17 below, which is necessary in the proof of proposition 2, and which, when generalized sufficiently, proves proposition 1 of section 2.7 as well. We begin with two definitions.

Definition 9 *Let D^3 be a region in $\mathbf{R}^3$ and let $\Sigma^2 = \partial D^3$ which we assume is smooth. Let*

$$
\begin{aligned}
D^3(x_1, x_2) &= \{(x,y,z) \in D^3 | x_1 < x < x_2\} \\
\Sigma^2(x_1, x_2) &= \{(x,y,z) \in \Sigma^2 | x_1 < x < x_2\} \\
C^2(x_1) &= \{(x,y,z) \in D^3 | x = x_1\}
\end{aligned}
$$

Definition 10 *Define $\Delta A(x_1, x_2)$ by*

$$\Delta A(x_1, x_2) = |C^2(x_1)| + |C^2(x_2)| + (36\pi)^{1/3} |D^3(x_1, x_2)|^{2/3} - |\Sigma^2(x_1, x_2)|$$

Hence, ΔA is the change in the surface area of D^3 if we cut a section of D^3 out from $x = x_1$ to $x = x_2$ and replace this section with a ball of equal volume.

Theorem 17 *There exist $\alpha, \beta > 0$ such that if for some $d > 0$*

1. $\inf_{0 \leq x \leq d} |C^2(x)| > 0$
2. $\frac{|\Sigma^2(0,d)|}{d^2} < \alpha$,

then there exists $x_1, x_2 \in [0, d], x_1 < x_2,$, such that

$$\frac{\Delta A(x_1, x_2)}{|\Sigma^2(x_1, x_2)|} < -\beta.$$

Proof. Since the theorem is scale-invariant, we may as well assume $d = 4$. Now we break the interval $[0, 4]$ into six intervals, the two at the ends having length 2ϵ and the four in the middle having length $1 - \epsilon$, for some positive $\epsilon \ll 1$. Let

$$\begin{aligned} I_1 &= [0, 2\epsilon] \\ I_2 &= [2\epsilon, 1 + \epsilon] \\ I_3 &= [1 + \epsilon, 2] \\ I_4 &= [2, 3 - \epsilon] \\ I_5 &= [3 - \epsilon, 4 - 2\epsilon] \\ I_6 &= [4 - 2\epsilon, 4] \end{aligned}$$

be these six intervals. Abusing notation slightly, let

$$p_k = \frac{|\Sigma^2(I_k)|}{|\Sigma^2(0, 4)|}, \quad 1 \leq k \leq 6,$$

so that $\sum_{k=1}^{6} p_k = 1$. Hence, p_k is the fraction of the area of $\Sigma^2(0, 4)$ which is in the region $I_k \times \mathbf{R}^2$.

Note that, in general, if

$$\frac{|\Sigma^2(x_1, x_2)|}{|\Sigma^2(0, 4)|} < \left(\frac{x_2 - x_1}{4}\right)^2, \tag{2.15}$$

then we can iterate this proof by substituting the interval (x_1, x_2) for $(0, 4)$ then rescaling $\mathbf{R}^3$ (in all directions) so that (x_1, x_2) becomes $(0, 4)$. Hypothesis 1 of the theorem is still satisfied, and hypothesis 2 is still satisfied too since areas scale as the square of distances. Hence, since the conclusion of the new rescaled theorem is stronger, the original theorem follows from the rescaled theorem.

We choose to rescale if inequality 2.15 is satisfied for $[x_1, x_2] = I_2, I_3, I_4, I_5, \cup_{k=1}^5 I_k$, or $\cup_{k=2}^6 I_k$. We claim this iteration process can only happen a finite number of times. First we note that

$$|\Sigma^2(0,4)| \geq \int_0^4 [\text{length of } \partial C^2(x)]\, dx \geq \int_0^4 \sqrt{4\pi |C^2(x)|}\, dx \geq 4\sqrt{4\pi a}$$

where we let $a = \inf_{0 \leq x \leq 4} |C^2(x)|$. Hence

$$a \leq \frac{|\Sigma^2(0,4)|^2}{64\pi}. \tag{2.16}$$

Each time we rescale, it follows from inequality 2.15 that the area of $\Sigma^2(0, 4)$ does not increase. However, a goes up by at least a factor of $\frac{4}{4-2\epsilon}$. Hence, since in the theorem we assumed $a > 0$, we must only rescale a finite number of times or inequality 2.16 would be violated.

In the final rescaled interval, we must therefore have

$$\frac{|\Sigma^2(x_1, x_2)|}{|\Sigma^2(0,4)|} \geq \left(\frac{x_2 - x_1}{4}\right)^2$$

for $[x_1, x_2] = I_2, I_3, I_4, I_5, \cup_{k=1}^5 I_k$,and $\cup_{k=2}^6 I_k$. Thus,

$$p_2, p_3, p_4, p_5 \geq \left(\frac{1-\epsilon}{4}\right)^2$$

and

$$\sum_{k=1}^5 p_k, \sum_{k=2}^6 p_k \geq \left(\frac{4-2\epsilon}{4}\right)^2.$$

Since $\sum_{k=1}^6 p_k = 1$, it follows (but it is not equivalent to) that

$$\begin{aligned} p_1, p_6 &\leq \epsilon && (2.17) \\ p_2, p_3, p_4, p_5 &\geq \frac{1}{16}(1-2\epsilon). && (2.18) \end{aligned}$$

To get an upper bound on $\frac{\Delta A(x_1,x_2)}{|\Sigma^2(x_1,x_2)|}$ in the conclusion of the theorem, we choose $x_1 = 0$, $x_2 = 4$ and determine the region D^3 which maximizes

$$\frac{\Delta A(0,4)}{|\Sigma^2(0,4)|} \tag{2.19}$$

while still satisfying 2.17 and 2.18.

This optimal region D^3 must be axially symmetric around the x-axis. This follows from the following symmetrization argument. Given a region D^3, symmetrize it about the x-axis by defining another region D^3_{SYM} to be axially symmetric around the x-axis but having the same cross sectional area as D^3 when intersected by planes given by x equal to a constant. D^3_{SYM} and D^3 have the same volume, and it is known that this symmetrization process decreases surface area. In fact, D^3_{SYM} will have less surface area than D^3 in each region $I_k \times \mathbf{R}^2$. We want to preserve inequalities 2.17 and 2.18, so define $\bar{D}^3$ to be D^3_{SYM} union any regions in $\mathbf{R}^3$ so that $\bar{D}^3$ has the same area as D^3 in each $I_k \times \mathbf{R}^2$. Then since $\bar{D}^3$ has more volume than D^3, we see that D^3 can only maximize 2.19 if it is axially symmetric.

Furthermore, from the first variation formula, $\Sigma^2 = \partial D^3$ must have constant mean curvature in each of the six intervals. Hence, in each interval Σ^2 is either a collection of spheres or a Delaunay surface. If α (from the statement of the theorem) is small enough, we can rule out Delaunay surfaces since they are unstable. We can also rule out more than one sphere completely contained in the interiors of each of the six intervals using stability since decreasing the area of one of the spheres while increasing the area of one of the other spheres at the same rate always increases volume to second order.

For convenience, let's rescale again so that $|\Sigma^2(0,d)| = 1$. Then checking all the possibilities we find that one of the optimal regions D^3 which maximizes $\Delta A(0,d)$

is the right portion (with outside surface area ϵ) of a ball in $I_1 \times \mathbf{R}^2$ union the left portion (with outside surface area ϵ) of a ball in $I_6 \times \mathbf{R}^2$ union a ball with surface area $\frac{1}{8}(1-2\epsilon)$ centered in $(I_2 \cup I_3) \times \mathbf{R}^2$ union a ball with surface area $\frac{7}{8}(1-2\epsilon)$ centered in $(I_4 \cup I_5) \times \mathbf{R}^2$. For this region D^3, we can then estimate that

$$|C^2(0)|, |C^2(d)| \leq \epsilon$$

and

$$|D^3(0,d)| \leq (36\pi)^{-1/2}\left[(2\epsilon)^{3/2} + (\frac{1}{8}(1-2\epsilon))^{3/2} + (\frac{7}{8}(1-2\epsilon))^{3/2}\right]$$

so that

$$\Delta A(0,d) \leq 2\epsilon + \left[(2\epsilon)^{3/2} + ((\frac{1}{8})^{3/2} + (\frac{7}{8})^{3/2})(1-2\epsilon)^{3/2}\right]^{2/3} - 1$$

where again we recall that we have rescaled so that $|\Sigma^2(0,d)| = 1$. Note that when $\epsilon = 0$, the right hand side of the above equation equals

$$\left[(\frac{1}{8})^{3/2} + (\frac{7}{8})^{3/2}\right]^{2/3} - 1 < 0.$$

Hence, by choosing ϵ small enough, we have

$$\frac{\Delta A}{|\Sigma^2(0,d)|} < -\beta$$

for some $\beta > 0$. Since this was for the maximal configuration for D^3, the theorem follows. $\square$

We call theorem 17 the "cutting theorem" since it tells us that if a region is long and skinny enough, then we can cut out a portion of it and replace that portion with a ball of equal volume and decrease the total surface area in the process. $\Delta A(x_1, x_2)$ is the amount the area changes when we cut out the section $D^3(x_1, x_2)$, and

$$\frac{|\Sigma^2(0,d)|}{d^2} < \alpha$$

is the condition we need to know that D^3 is long and skinny enough.

Intuitively, this theorem is clear, but we see that the proof was nontrivial. We claim, but neglect to prove here, two generalizations of theorem 17. First, we will need a cutting theorem like theorem 17 for the Schwarzschild metric outside a coordinate ball of radius R to prove proposition 2, for some large $R > 0$. Since the Schwarzschild metric $(\mathbf{R}^3 - \{0\}, h)$ has conformal factor $(1+\frac{m}{2r})^4$ which approaches 1 as r approaches infinity, we can view the Schwarzschild metric as a perturbation of $\mathbf{R}^3$, with the perturbation being as small as we like if we choose R large enough. Hence, it is reasonable to use rotated and translated versions of the standard $\mathbf{R}^3 - \{0\}$ coordinate chart for the Schwarzschild metric of mass m to define $D^3(x_1, x_2)$, $\Sigma^2(x_1, x_2)$, and $C^2(x_1)$, and then to state a generalized version of theorem 17 for regions D^3 in the Schwarzschild metric entirely outside the coordinate ball of radius R, for some $R > 0$. Secondly, we claim that proposition 1 follows as a further generalization of theorem 17, where not only are we now in the Schwarzschild metric of mass m, but the cuts are being made along planes parallel to the sides of a large polyhedron contained inside the coordinate ball of radius r_2 minus the coordinate ball of radius r_1. In this way proposition 1 follows, after sufficient adaptation, from the proof of theorem 17.

Now we prove proposition 2 from section 2.7. Since the Schwarzschild metric is conformal to $\mathbf{R}^3 - \{0\}$, with conformal factor $(1 + \frac{m}{2r})^4$, then for surfaces outside the coordinate ball of radius $\tilde{r}$ we can use the isoperimetric inequality for $\mathbf{R}^3$ to conclude that

$$A^{3/2} \geq \sqrt{36\pi} V (1 + \frac{m}{2\tilde{r}})^{-6}.$$

Let $A(V_S, V)$ be the area of a constant curvature 2-sphere containing a volume V in the constant curvature 3-sphere of volume V_S. Then since $A(V_S, V)^{3/2} < \sqrt{36\pi} V$ for $V > 0$, with the inequality being by a uniform amount for $V \geq \epsilon$ given an $\epsilon > 0$, we see that we can simply choose $\tilde{r}$ large enough to prove proposition 2 for $V \geq \epsilon$.

To prove proposition 2 for small V, we observe that

$$A(V_S, V)^{3/2} = \sqrt{36\pi} V \left[1 - k \left(\frac{V}{V_S} \right)^{2/3} + O_2[\left(\frac{V}{V_S} \right)^{2/3}] \right] \tag{2.20}$$

where $k = \frac{3}{10} \left(\frac{3\pi}{2} \right)^{2/3}$. We will show that for small volumes $V < \epsilon$, if we choose $\tilde{r}$

large enough, then in $(\mathbf{R}^3 - \{0\}, h)$ outside the coordinate ball of radius $\tilde{r}$ that all surfaces of area A containing a volume V satisfy

$$A^{3/2} \geq \sqrt{36\pi} V \left[1 - \frac{1}{2} k \left(\frac{V}{V_S} \right)^{2/3} \right] \tag{2.21}$$

which will prove proposition 2 for $V < \epsilon$ if we choose ϵ small enough. Thus, all that remains is to establish inequality 2.21 for $V < \epsilon$, for some $\epsilon > 0$.

Suppose $\Sigma^2 = \partial D^3$ contains a volume $V = |D^3| < \epsilon$ and is entirely outside the coordinate ball of radius $\tilde{r}$ in Schwarzschild. We assume Σ^2 is smooth, but Σ^2 could have tentacles extending long distances, for example, which is troublesome. We find it necessary to regularize Σ^2 first before proving inequality 2.21.

Let U^3 be any open subset of the bounded open set D^3. Define

$$f(U^3) = \text{Area}(\partial(D^3 - U^3)) + (36\pi)^{1/3} \text{Volume}(U^3)^{2/3}.$$

Since we have uniform bounds on $|U^3|$ and $|\partial U^3|$ when f is being minimized since $|\partial U^3| \leq |\partial(D^3 - U^3)| + |\partial D^3|$, and since D^3 is bounded, there exists a region $U_0 \subset D^3$ which minimizes f. Note that since $f(\emptyset) = |\partial D^3|$, $f(U_0) \leq |\partial D^3|$.

Finally, we regularize D^3 by removing the region U_0 from D^3. This, of course, decreases the total volume, so to keep the total volume constant we add a ball of volume $|U_0|$ to a copy of $\mathbf{R}^3$. Thus, we've modified D^3 and replaced it with $\bar{D}^3 = (D^3 - U_0) \cup B^3 \subset \text{Schwarzschild} \cup \mathbf{R}^3$, where B^3 is the ball of volume $|U_0|$ in $\mathbf{R}^3$. Note that $|\bar{D}^3| = |D^3|$ and that the area has decreased since

$$|\partial \bar{D}^3| = |\partial(D^3 - U_0)| + (36\pi)^{1/3} |U_0|^{2/3} = f(U_0) \leq f(\emptyset) = |\partial D^3|.$$

Thus, it is sufficient to prove inequality 2.21 for the regularized region $\bar{D}^3$ in Schwarzschild (disjoint) union $\mathbf{R}^3$.

It is also sufficient to prove inequality 2.21 for each component of $\bar{D}^3$ individually. The ball B^3 in $\mathbf{R}^3$ satisfies inequality 2.21 since $A^{3/2} = \sqrt{36\pi} V$ for balls. Now consider one of the components $\Sigma_i^2 = \partial \bar{D}_i^3$ in Schwarzschild outside the coordinate ball of radius $\tilde{r}$. Note that for Σ_i^2,

$$A \leq (36\pi)^{1/3} V^{2/3} \tag{2.22}$$

because otherwise U_0 would have included the region $\bar{D}_i^3$. Furthermore,

$$\mathrm{diam}(\Sigma_i^2) \leq \alpha^{-1/2}(36\pi)^{1/6} V^{1/3}, \tag{2.23}$$

where α is the constant from the cutting theorem and $\mathrm{diam}(S)$ is the diameter of S. Otherwise, we would have $\mathrm{diam}(\Sigma_i^2) > \alpha^{-1/2}((36\pi)^{1/3}V^{2/3})^{1/2} \geq \alpha^{-1/2}A^{1/2}$ which means we could use the cutting theorem to remove a section of Σ_i^2, form a ball in $\mathbf{R}^3$ with it, and decrease the boundary area while preserving the total volume. This cannot happen, since U_0 would have included this section of Σ_i^2 if removing it and forming a ball with it decreased the total area. Hence, we must have inequality 2.23. This diameter bound is central to the rest of the proof and is the reason we needed to regularize D^3.

Pick any point p_0 in $\bar{D}_i^3$, where again $\Sigma_i^2 = \partial\bar{D}_i^3$. In coordinates, Schwarzschild can be represented as $(\mathbf{R}^3 - \{0\}, h)$, where $h_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}$ is the metric and r is the radial coordinate in $\mathbf{R}^3$. Suppose p_0 has radial coordinate r_0. Since Σ_i^2 is outside the coordinate ball of radius $\tilde{r}$, $r_0 \geq \tilde{r}$.

We construct a spherically-symmetric mapping ϕ from a spherically-symmetric connected neighborhood of Schwarzschild containing p_0 to a spherically-symmetric connected annular neighborhood of a large 3-sphere S^3 of radius R_0 (when embedded in $\mathbf{R}^4$). We want $\phi : (\mathbf{R}^3 - \{0\}, h) \to (S^3, g_0)$ to be spherically-symmetric, locally volume preserving, and "tangent" (to be defined in a moment) at p_0.

Let

$$u(r)^{-1} = \frac{\|D\phi(\partial_r)\|_{g_0}}{\|\partial_r\|_h}$$

where $\partial_r = \frac{\partial}{\partial r}$ is a radial tangent vector in Schwarzschild, $\|\cdot\|_h$ is the length in the Schwarzschild metric, and $\|\cdot\|_{g_0}$ is the length in the sphere of radius R_0 metric. Thus ϕ increases lengths in the radial direction by a factor of $u(r)^{-1}$. Since ϕ preserves volumes locally, lengths in the two other mutually orthogonal directions must be increased by a factor of $u(r)^{1/2}$, so that the areas of the spherically symmetric spheres of the Schwarzschild metric get increased by a factor of $u(r)$ by ϕ. We choose the radius R_0 of S^3 and define ϕ such that $u(r_0) = 1$ and $\frac{du}{dr}(r_0) = 0$, in which case we say $\phi : (\mathbf{R}^3 - \{0\}, h) \to (S^3, g_0)$ is tangent at $r = r_0$, and in particular at p_0.

Since volume is preserved by ϕ locally, it is most convenient to parameterize the spherically symmetric functions by the enclosed volume of the corresponding spherically symmetric spheres, or at least relative enclosed volume. On Schwarzschild, define $v(r)$ to be the volume enclosed by the coordinate ball of radius r outside the coordinate ball of radius r_0. When $r < r_0$, $v(r) < 0$, and $v(r_0) = 0$. Let $U(v)$ be $u(r)$ changed into v-coordinates, and let $A_0(v)$ be the area of the spherically symmetric sphere in Schwarzschild containing a volume v outside the coordinate ball of radius r_0. Use ϕ to define v on (S^3, g_0), and let $A_1(v)$ be the area of the spherically symmetric spheres in (S^3, g_0). Since ϕ is locally volume preserving, v is relative enclosed volume on (S^3, g_0) as well as $(\mathbf{R}^3 - \{0\}, h)$.

In the Schwarzschild metric of mass m,

$$\left(\frac{A_0(v)}{16\pi}\right)^{1/2}\left(1 - \frac{1}{16\pi}A_0(v)A_0'(v)^2\right) = m$$

for all v. This follows from the fact that the mean curvature of the spheres is given by $H = A'(v)$ and the formula for the Hawking mass. In a 3-sphere (S^3, g_0) of radius R_0 (when embedded in $\mathbf{R}^4$), we compute directly that

$$\frac{4\pi}{A_1(v)}\left(1 - \frac{1}{16\pi}A_1(v)A_1'(v)^2\right) = R_0^{-2}.$$

At the point of tangency ($v = 0$), $A_0(0) = A_1(0)$ and $A_0'(0) = A_1'(0)$. Hence, dividing the two previous formulas at $v = 0$ gives us that

$$mR_0^2 = \frac{1}{2}\left(\frac{A_0(0)}{4\pi}\right)^{3/2}.$$

Hence, the further out p_0 is in Schwarzschild, the larger $A_0(0)$ is and the larger R_0 is. Thus, we may guarantee R_0 to be as large as we like if we choose $\tilde{r}$ large enough.

(Also, we see that this construction only works when $m > 0$, which, by the positive mass theorem, is all we need. Proposition 2 is true for $m \leq 0$, but the proof requires constructing tangent hyperbolic spaces instead of tangent spheres.)

Furthermore, since $A_1(v) = U(v)A_0(v)$, we can differentiate this twice, and use $A_0(0) = A_1(0)$ and $A_0'(0) = A_1'(0)$ to get

$$U''(0) = \frac{A_1''(0) - A_0''(0)}{A_0(0)}.$$

Changing the $U(v)$ back to r coordinates, we get

$$u''(r_0) = A_0(0)(A_1''(0) - A_0''(0))$$

where the derivatives on A_0 and A_1 are still with respect to v. Working out the behavior of $A_0''(0)$ and $A_1''(0)$ for large $A_0(0)$, we find that $u''(r_0)$ goes down as $k/A_0(0)$, for some constant k for large $A_0(0)$. Thus, we can guarantee $u''(r_0)$ to be as small as we like if we choose $\tilde{r}$ large enough.

Since $V < \epsilon$, $\operatorname{diam}(\Sigma_i^2) \leq \alpha^{-1/2}(36\pi)^{1/6}\epsilon^{1/3}$, so we need only to extend ϕ this distance both ways from p_0. Hence, on Σ_i^2,

$$u(r) \geq 1 - \delta(r - r_0)^2$$

for some $\delta > 0$ since $u(r_0) = 1$ and $u'(r_0) = 0$, and we can choose δ as small as we want if we choose $\tilde{r}$ large enough since this will make $u''(r_0)$ small. Thus, since direct calculation shows that $u(r) \leq 1$, ϕ increases areas pointwise by a factor less than or equal to $u(r)^{-1/2}$, and

$$\begin{aligned} u(r) &\geq 1 - \delta\left[\operatorname{diam}(\Sigma_i^2)\right]^2 \\ &\geq 1 - \delta\alpha^{-1}(36\pi)^{1/3}V^{2/3}. \end{aligned}$$

We will use the isoperimetric inequality for (S^3, g_0), that the spherically symmetric spheres minimize area among surfaces enclosing the same volume, to prove inequality 2.21 for $V < \epsilon$. If we choose $\tilde{r}$ large enough, the radius R_0 (and total volume) of

(S^3, g_0) will be as large as we want, so by inequality 2.20

$$A^{3/2} \geq \sqrt{36\pi} V[1 - \delta V^{2/3}] \text{ in } (S^3, g_0)$$

for $V < \epsilon$. Since $\phi : (\mathbf{R}^3 - \{0\}, h) \to (S^3, g_0)$ increases areas less than $u(r)^{-1/2}$, then in Schwarzschild we have

$$\begin{aligned} A^{3/2} &\geq u(r)^{3/4} \sqrt{36\pi} V[1 - \delta V^{2/3}] \\ &\geq \sqrt{36\pi} V[1 - \delta V^{2/3}][1 - \delta \alpha^{-1} (36\pi)^{1/3} V^{2/3}]^{3/4} \end{aligned}$$

for $V < \epsilon$ if we choose $\tilde{r}$ large enough. Since $\delta > 0$ could be chosen as small as we like provided $\tilde{r}$ was chosen large enough, inequality 2.21 follows, proving the theorem. □

2.10 Conjectures

We have seen that isoperimetric surfaces can be used to prove the Penrose inequality for two classes of manifolds. Naturally we want to generalize these results. First we minimized area with a volume constraint and found that this approach worked as long as the minimizing surfaces always had only one component. Then we realized that if we minimized the sum of the areas to the three halves power with a volume constraint, then this approach worked even when the minimizer had multiple components. However, this second approach has a new problem, that two or more surfaces in the minimizing configuration can push up against each other.

This suggests that we are still not optimizing the correct quantity. Minimizing the sum of the areas to the three halves power is a generalization of minimizing area (with a volume constraint) in the sense that these two optimization problems give the same answer when the minimizers have only one component. Hence, it is natural to consider how we can generalize the quantity "sum of the areas to the three halves power" in such a way that the new quantity equals the sum of the areas to the three halves power in certain cases. We recall the definition of area nonincreasing maps given in definition 8 of section 2.6 and propose the following functional.

Definition 11 *Suppose (M^3, g) is complete, has nonnegative scalar curvature, contains one or more outermost minimal spheres $\{\Sigma_i\}$, and is asymptotically flat at infinity. Let*

$$f(D^3) = \sup_{\phi}\{ Vol_{\mathbf{R}^3}(\phi(D^3)) \mid \phi : D^3 \to \mathbf{R}^3 \textit{is area nonincreasing}\}$$

and

$$f(V) = \inf_{D^3}\{f(D^3) \mid D^3 \textit{contains a volume } V \textit{ outside the horizons}\{\Sigma_i\}\}$$

where D^3 is any open region in M^3 containing everything inside the horizons.

In general, there will not be a unique map ϕ which maximizes the $\mathbf{R}^3$ volume of $\phi(D^3)$ among area nonincreasing maps. Instead, we generally expect there to be a two dimensional "critical set" of D^3 which determines how large the $\mathbf{R}^3$ volume of $\phi(D^3)$ can be. For example, if the boundary of D^3 has area A, then by the isoperimetric inequality in $\mathbf{R}^3$, the volume of $\phi(D^3)$ can be at most the volume of a sphere in $\mathbf{R}^3$ with surface area A, which is $A^{\frac{3}{2}}/\sqrt{36\pi}$, and sometimes this upper bound is realized. In fact, if D^3 has several components each with boundary area A_i, then the volume of $\phi(D^3)$ can be at most the volume of disjoint balls with surface areas A_i, which is $\sum A_i^{\frac{3}{2}}/\sqrt{36\pi}$, and sometimes this upper bound is realized too. Hence, we see that optimizing the functional f is sometimes equivalent to optimizing the sum of the areas to the three halves power.

Conjecture 2 *Under the assumptions stated in the definition of $f(V)$,*

$$f''(V) \leq \frac{1 - f'(V)^2}{6f(V)}$$

Definition 12 *For $V \geq 0$, let*

$$m(V) = f(V)^{\frac{1}{3}}(1 - f'(V)^2)/k$$

be the new mass function, where $k = (32\pi/3)^{\frac{1}{3}}$.

From conjecture 2, it follows that $m(V)$ is nondecreasing for $V \geq 0$. Also, the original manifold can always be modified so that $m(0) = \left(\sum_{i=1}^{n} \left(\frac{A_i}{16\pi}\right)^{\frac{3}{2}}\right)^{\frac{1}{3}}$ where $\{A_i\}$ are the areas of the horizons. Hence, it is likely that conjecture 2 would imply the following generalized Penrose inequality.

Conjecture 3 *Suppose* (M^3, g) *is complete, has nonnegative scalar curvature, contains one or more outermost minimal spheres* $\{\Sigma_i\}$ *with surface areas* $\{A_i\}$*, and is Schwarzschild with mass* m *at infinity. Then* $m \geq \left(\sum_{i=1}^{n} \left(\frac{A_i}{16\pi}\right)^{\frac{3}{2}}\right)^{\frac{1}{3}}$.

Chapter 3

Volume Comparison Theorems

The isoperimetric surface techniques which we developed to study the Penrose inequality in general relativity also can be used to prove several volume comparison theorems, including a new proof of Bishop's volume comparison theorem for positive Ricci curvature. Let (S^n, g_0) be the standard metric (with any scaling) on S^n with constant Ricci curvature $Ric_0 \cdot g_0$. Bishop's theorem says that if (M^n, g) is a complete Riemannian manifold ($n \geq 2$) with $Ric(g) \geq Ric_0 \cdot g$, then $\mathrm{Vol}(M^n) \leq \mathrm{Vol}(S^n)$. It is then natural to ask whether a similar type of volume comparison theorem could be true for scalar curvature. We prove the following theorems for 3-manifolds.

Theorem 18 *Let (S^3, g_0) be the constant curvature metric on S^3 with scalar curvature R_0, Ricci curvature $Ric_0 \cdot g_0$, and volume V_0. Then there exists a positive $\epsilon_0 < 1$ such that if (M^3, g) is any complete smooth Riemannian manifold of volume V satisfying*

$$R(g) \geq R_0$$

$$Ric(g) \geq \epsilon_0 \cdot Ric_0 \cdot g$$

then

$$V \leq V_0.$$

As it happens, a lower bound on scalar curvature by itself is not sufficient to give an upper bound on the total volume. We can scale a cylinder, $S^2 \times \mathbf{R}$, to have any positive scalar curvature and still have infinite volume. However, if we consider a neighborhood of metrics around the standard metric g_0 on S^3 which satisfy $Ric(g) \geq \epsilon_0 \cdot Ric_0 \cdot g$, then from the above theorem we see that $R(g) \geq R_0$ implies that $V \leq V_0$ for these metrics. Hence, we see that a volume comparison theorem for scalar curvature is true for metrics close to the standard metric on S^3. Moreover,

Theorem 19 *Let (S^3, g_0) be the constant curvature metric on S^3 with scalar curvature R_0, Ricci curvature $Ric_0 \cdot g_0$, and volume V_0. If $\epsilon \in (0,1]$ and (M^3, g) is any complete smooth Riemannian 3-manifold of volume V satisfying*

$$R(g) \geq R_0$$

$$Ric(g) \geq \epsilon \cdot Ric_0 \cdot g$$

then

$$V \leq \alpha(\epsilon) V_0$$

where

$$\alpha(\epsilon) = \sup_{\frac{4\pi}{3-2\epsilon} \leq z \leq 4\pi} \frac{1}{\pi^2} \begin{pmatrix} \int_0^{y(z)} \left(36\pi - 27(1-\epsilon) y(z)^{\frac{2}{3}} - 9\epsilon \cdot x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx \\ + \int_{y(z)}^{z^{\frac{3}{2}}} (36\pi - 18(1-\epsilon) y(z) x^{-\frac{1}{3}} - 9x^{\frac{2}{3}})^{-\frac{1}{2}} dx \end{pmatrix}$$

where

$$y(z) = \frac{z^{\frac{1}{2}}(4\pi - z)}{2(1-\epsilon)}.$$

Furthermore, this expression for $\alpha(\epsilon)$ is sharp.

Interestingly enough, $\alpha(\epsilon) = 1$ for many values of ϵ. And since the above expression for $\alpha(\epsilon)$ is sharp, this allows us to define the best value for ϵ_0 which works in theorem 18, namely

$$\epsilon_0 = \inf\{\epsilon \in (0,1] \mid \alpha(\epsilon) = 1\}$$

Naturally it is desirable to estimate the actual value of ϵ_0. It is not too hard to show that $\epsilon_0 < 1$. However, getting an accurate estimate for ϵ_0 definitely seems to be a job for a computer, and it seems reasonable to conjecture that ϵ_0 is transcendental.

From preliminary computer calculations, it looks like $0.134 < \epsilon_0 < 0.135$, although these bounds are not currently rigorous.

3.1 Isoperimetric Surface Techniques

As before in chapter 2, isoperimetric surfaces will be used to prove these theorems. The main difference is that we will be minimizing area with a volume constraint on compact manifolds in this chapter, so existence of area minimizers is already known. Also, the manifolds we will be considering all have positive Ricci curvature, from which it will follow from a stability argument that the area minimizers always have exactly one component. Hence, condition 1 from chapter 2 will always apply, so it will not be necessary to consider minimizing F with a volume constraint.

Definition 13 *Let (M^n, g) be a complete Riemannian n-manifold. Define*

$$A(V) = \inf_R \{Area(\partial R) \mid Vol(R) = V\}$$

where R is any region in M^n, $Vol(R)$ is the n dimensional volume of R, and $Area(\partial R)$ is the $n-1$ dimensional volume of ∂R. If there exists a region R with $Vol(R) = V$ such that $Area(\partial R) = A(V)$, then we say that $\Sigma = \partial R$ minimizes area with the given volume constraint.

The manifolds we will be dealing with in this chapter all have $Ric(g) \geq \delta > 0$. Hence, these manifolds are compact, so there will always exist a minimizer $\Sigma(V)$ (not necessarily unique) for all V. These minimal surfaces have constant mean curvature and are smooth.

We will use the function $A(V)$ to achieve the volume bounds on M^n. We will use the curvature bounds on M^n to get an upper bound on $A''(V)$. Intuitively, this will force the two roots of $A(V)$ to be close together. Since the two roots of $A(V)$ are 0 and Vol(M^n), we will get an upper bound for Vol(M^n).

To get an upper bound for $A''(V)$ at $V = V_0$, we will do a unit normal variation on $\Sigma(V_0)$. That is, let $\Sigma_{V_0}(t)$ be the surface created by flowing $\Sigma(V_0)$ out at every point in the normal direction at unit speed for time t. Since $\Sigma(V_0)$ is smooth, we can do this variation for $t \in (-\delta, \delta)$ for some $\delta > 0$. Abusing notation slightly, we can also parameterize these surfaces by their volumes as $\Sigma_{V_0}(V)$ so that $V = V_0$ will correspond to $t = 0$. Let $A_{V_0}(V) = \text{Area}(\Sigma_{V_0}(V))$. Then $A(V_0) = A_{V_0}(V_0)$ and $A(V) \leq A_{V_0}(V)$ since $\Sigma_{V_0}(V)$ is not necessarily minimizing for its volume. Hence,

$$A''(V_0) \leq A''_{V_0}(V_0).$$

Let us suppose that (M^3, g) satisfies $R(g) \geq R_0$ and $Ric(g) \geq \epsilon \cdot Ric_0 \cdot g$ as in theorem 19 and compute $A''_{V_0}(V_0)$. To do this, we will need to compute the first and second derivatives of the area of $\Sigma_{V_0}(t)$ and the volume that it encloses. We will use the formulas

$$\dot{d\mu} = H\, d\mu \quad \text{and} \quad \dot{H} = -||\Pi||^2 - Ric(\nu, \nu) \tag{3.1}$$

where the dot represents differentiation with respect to t, $d\mu$ is the surface area 2-form for $\Sigma_{V_0}(t)$, Π is the second fundamental form for $\Sigma_{V_0}(t)$, $H = \text{trace}(\Pi)$ is the mean curvature, and ν is the outward pointing unit normal vector. Since $A_{V_0}(t) = \int_{\Sigma_{V_0}(t)} d\mu$,

$$A'_{V_0}(t) = \int_{\Sigma_{V_0}(t)} H\, d\mu$$

And since $V'(t) = \int_{\Sigma_{V_0}(t)} d\mu = A_{V_0}(t)$,

$$A'_{V_0}(V_0) = A'_{V_0}(0)/V'(0) = H$$

By single variable calculus,

$$A''_{V_0}(V) = \frac{A''_{V_0}(t) - A'_{V_0}(V)V''(t)}{V'(t)^2}$$

so that at $t = 0$,

$$\begin{aligned} A_{V_0}(V_0)^2 A''_{V_0}(V_0) &= A''_{V_0}(t) - HV''(t) \\ &= \frac{d}{dt}\int_{\Sigma_{V_0}(t)} H\, d\mu \;-\; H\frac{d}{dt}\int_{\Sigma_{V_0}(t)} d\mu \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Sigma(V_0)} \dot{H}\, d\mu \\
&= \int_{\Sigma(V_0)} -||\Pi||^2 - Ric(\nu,\nu)
\end{aligned}
$$

Finally, since $||\Pi||^2 \geq \frac{1}{2}\mathrm{trace}(\Pi)^2 = \frac{1}{2}H^2$ and $Ric(\nu,\nu) \geq \epsilon \cdot Ric_0$,

$$
\begin{aligned}
A_{V_0}(V_0)^2 A''_{V_0}(V_0) &\leq \int_{\Sigma(V_0)} -\frac{1}{2}H^2 - \epsilon \cdot Ric_0 \\
&= -A_{V_0}(V_0)\left(\frac{1}{2}A'_{V_0}(V)^2 + \epsilon \cdot Ric_0\right)
\end{aligned}
$$

Hence,

$$A''_{V_0}(V_0) \leq -\frac{1}{A_{V_0}(V_0)}\left(\frac{1}{2}A'_{V_0}(V_0)^2 + \epsilon \cdot Ric_0\right) \tag{3.2}$$

Lemma 4 *Suppose $\Sigma = \partial R$ minimizes area for its volume, $R \subset (M^3, g)$, and $Ric(g) \geq \delta > 0$. Then Σ has exactly one component.*

Proof. Suppose Σ has more than one component. Consider a flow on Σ which is a unit normal flow (flowing out) on the first component (parameterized by volume) and a unit normal flow (flowing in) on the second component (also parameterized by volume). Then all of the surfaces in this family contain the same volume. However, by equation 3.2 the second derivative of area is negative with respect to this volume preserving flow (let $\delta = \epsilon \cdot Ric_0$). Thus, Σ does not minimize area for its volume. Contradiction. □

Lemma 4 will be crucial for getting upper bounds on $A''(V)$ from the lower bound on scalar curvature, and is one of the reasons we need some kind of lower bound on Ricci curvature.

Going back to equation 3.2, since $A(V_0) = A_{V_0}(V_0)$ and $A(V) \leq A_{V_0}(V)$,

$$A''(V) \leq -\frac{1}{A(V)}\left(\frac{1}{2}A'(V)^2 + \epsilon \cdot Ric_0\right) \tag{3.3}$$

in the sense of comparison functions defined in chapter 2.

Lemma 5 *Suppose (M^3, g) satisfies $Ric(g) \geq \delta > 0$. Then $A(V)$ is strictly increasing on the interval $[0, \frac{1}{2}\, Vol(M^3)]$.*

Proof. It is always true that $A(V) = A(\mathrm{Vol}(M^3) - V)$, since the boundaries of a region and its complement are the same. By equation 3.3, $A''(V)$ is strictly negative (again, let $\delta = \epsilon \cdot Ric_0$). The lemma follows. $\square$

Now we want an equation like equation 3.3 which follows from the lower bound on scalar curvature. From before,

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} -||\Pi||^2 - Ric(\nu, \nu)$$

By the Gauss equation,

$$Ric(\nu, \nu) = \frac{1}{2}R - K + \frac{1}{2}H^2 - \frac{1}{2}||\Pi||^2$$

where R is the scalar curvature of M^3 and K is the Gauss curvature of $\Sigma(V_0)$. Substituting we get,

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} -\frac{1}{2}R + K - \frac{1}{2}H^2 - \frac{1}{2}||\Pi||^2$$

By Lemma 4, $\Sigma(V_0)$ has only one component, so by the Gauss-Bonnet theorem, $\int_{\Sigma(V_0)} K = 2\pi X(\Sigma(V_0)) \leq 4\pi$. Since $R \geq R_0$ and $||\Pi||^2 \geq \frac{1}{2}H^2$, we have

$$\begin{aligned} A_{V_0}(V_0)^2 A''_{V_0}(V_0) &\leq 4\pi - \int_{\Sigma(V_0)} \frac{1}{2}R_0 + \frac{3}{4}H^2 \\ &= 4\pi - A_{V_0}(V_0)\left(\frac{1}{2}R_0 + \frac{3}{4}H^2\right) \end{aligned}$$

Hence,

$$A''_{V_0}(V_0) \leq \frac{4\pi}{A_{V_0}(V_0)^2} - \frac{1}{A_{V_0}(V_0)}\left(\frac{3}{4}A'_{V_0}(V_0)^2 + \frac{1}{2}R_0\right)$$

As before, since $A(V_0) = A_{V_0}(V_0)$ and $A(V) \leq A_{V_0}(V)$,

$$A''(V) \leq \frac{4\pi}{A(V)^2} - \frac{1}{A(V)}\left(\frac{3}{4}A'(V)^2 + \frac{1}{2}R_0\right) \tag{3.4}$$

in the sense of comparison functions defined in chapter 2.

Notice that we had to use the Gauss-Bonnet theorem to get equation 3.4. If we tried to generalize equation 3.4 for higher dimensions, we would need to get an upper bound for $\int_{\Sigma(V_0)} R^\Sigma$, where R^Σ is the scalar curvature of $\Sigma(V_0)$. Since we don't have such a bound in general, the argument, as presented here, only works when M is a 3-manifold.

However, equation 3.3 does generalize for all dimensions. This allows us to give a new proof of Bishop's theorem, which we present in section 3.3.

3.2 Ricci and Scalar Curvature Mass

We define

$$F(V) = A(V)^{\frac{3}{2}} \tag{3.5}$$

and choose to deal with $F(V)$ instead of $A(V)$. Since $F(V)$ and V have the same units and $F(V)$ is roughly a linear function of V for small V, the equations for $F(V)$ turn out to be simpler than the equations for $A(V)$. Of course, $F(V)$ and $A(V)$ will have the same roots, 0 and $\mathrm{Vol}(M^3)$, and we will want to use upper bounds on $F''(V)$ to prove that the roots of $F(V)$ are close together, thereby getting an upper bound on $\mathrm{Vol}(M^3)$.

Plugging equation 3.5 into equations 3.3 and 3.4 and simplifying, we get

$$F''(V) \leq -\frac{3\epsilon \cdot Ric_0}{2} F(V)^{-\frac{1}{3}} \tag{3.6}$$

and

$$F''(V) \leq \frac{36\pi - F'(V)^2}{6F(V)} - \frac{3R_0}{4} F(V)^{-\frac{1}{3}} \tag{3.7}$$

in the sense of comparison functions defined in chapter 2. We comment that it follows that these inequalities are also true distributionally. Given inequalities like equations 3.6 and 3.7, it is natural to want to integrate them.

Definition 14 *Let*

$$m_{Ric}(V) = \left(36\pi - F'(V)^2\right) - \frac{9\epsilon \cdot Ric_0}{2} F(V)^{\frac{2}{3}}$$

$$m_R(V) = F(V)^{\frac{1}{3}} \left(36\pi - F'(V)^2\right) - \frac{3R_0}{2} F(V)$$

and we call these two quantities "Ricci curvature mass" and "scalar curvature mass" respectively.

$F(V)$ is continuous, but $F'(V)$ does not necessarily exist for all V, although it does exist almost everywhere since $F(V)$ is monotone increasing on $[0, \frac{1}{2}\mathrm{Vol}(M^3)]$ and monotone decreasing on $[\frac{1}{2}\mathrm{Vol}(M^3), \mathrm{Vol}(M^3)]$. The left and right hand derivatives, $F'_+(V)$ and $F'_-(V)$, do always exist though. This follows from the fact that $F(V)$ has comparison functions $F_{V_0}(V) = A_{V_0}(V)^{\frac{3}{2}}$, for all $V_0 \in (0, \mathrm{Vol}(M^3))$ with uniformly bounded second derivatives. Hence, we can add a quadratic to $F(V)$ to get a concave function, from which it follows that the left and right hand derivatives exist and are equal except at a countable number of points. We define $F'_-(0) = \sqrt{36\pi}$ and $F'_+(\mathrm{Vol}(M^3)) = -\sqrt{36\pi}$, which is natural for smooth manifolds.

Furthermore, $F'_+(V) \leq F'_-(V)$ using the comparison function argument again since $F'_+(V_0) \leq F'_{V_0}(V_0) \leq F'_-(V_0)$. If $F'(V)$ does not exist, then it is natural to define $F'(V)$ to be a multivalued function taking on every value in the interval $(F'_+(V), F'_-(V))$. This is consistent, since if $F'(V)$ does exist, then $F'_+(V) = F'_-(V)$. Hence, $m_{Ric}(V)$ and $m_R(V)$ are multivalued for some V, which can be interpreted as the mass "jumping up" at these V, and the set of V for which $m(V)$ and $F(V)$ are multivalued is a countable set. Alternatively, one could replace $F'(V)$ with $F'_+(V)$ (or $F'_-(V)$) in the formulas for $m_{Ric}(V)$ and $m_R(V)$ so that they would always be single valued.

Lemma 6 *The quantities $m_{Ric}(V)$ and $m_R(V)$ are nonnegative, nondecreasing functions of V on the interval $[0, \frac{1}{2}\mathit{Vol}(M^3)]$ and $m_{Ric}(0) = m_R(0) = 0$.*

Proof. Since M^3 is a smooth manifold, $F(V) \approx \sqrt{36\pi}V$ for small V and $F'(0) = \sqrt{36\pi}$. Since $F(0) = 0$, it follows that $m_{Ric}(0) = m_R(0) = 0$. In addition, we observe that if $F(V)$ were smooth, then

$$m'_{Ric}(V) = 2F'(V)\left(-F''(V) - \frac{3\epsilon \cdot Ric_0}{2}F^{-\frac{1}{3}}\right)$$

and

$$m'_R(V) = 2F^{\frac{1}{3}}F'(V)\left(-F''(V) + \frac{36\pi - F'(V)^2}{6F(V)} - \frac{3R_0}{4}F^{-\frac{1}{3}}\right)$$

Then by lemma 5, we would have $F'(V) \geq 0$, so that by equations 3.6 and 3.7, $m'_{Ric}(V) \geq 0$ and $m'_R(V) \geq 0$ proving that $m_{Ric}(V)$ and $m_R(V)$ are nondecreasing, and hence nonnegative, on the interval $[0, \frac{1}{2}\text{Vol}(M^3)]$.

More generally, we need to prove that $m'_{Ric}(V) \geq 0$ and $m'_R(V) \geq 0$ as distributions, which follows as before in the proof of lemma 1.□

The reason the we call the two quantities m_{Ric} and m_R "mass" is motivated by the fact that if we set $R_0 = 0$, $m_R(V) = m(V)$, where $m(V)$ is the mass function from chapter 2. However, beyond being nonnegative, nondecreasing functions which are very similar to $m(V)$ in form, the author is not currently aware of any physical interpretations of $m_{Ric}(V)$ and $m_R(V)$ in the context of general relativity, although that is an interesting possibility.

3.3 A New Proof of Bishop's Theorem

In this section, we will give a new proof of Bishop's theorem using an argument which is very similar to the one we will use to prove theorem 19. Whereas the rest of this chapter deals specifically with 3-manifolds, in the next section we will study n-manifolds. Because of this, we will need to generalize a few definitions and equations just for this section.

Theorem 20 (Bishop) *Let (S^n, g_0) be the standard metric (with any scaling) on S^n with constant Ricci curvature $Ric_0 \cdot g_0$. If (M^n, g) is a complete Riemannian manifold ($n \geq 2$) with $Ric(g) \geq Ric_0 \cdot g$, then Vol$(M^n) \leq$ Vol(S^n).*

Proof. Modifying equation 3.3, since $||\Pi||^2 \geq \frac{1}{n-1}\text{trace}(\Pi)^2 = \frac{1}{n-1}H^2$, $(\epsilon = 1)$

$$A''(V) \leq -\frac{1}{A(V)}\left(\frac{1}{n-1}A'(V)^2 + Ric_0\right) \tag{3.8}$$

Now we let $F(V) = A(V)^{\frac{n}{n-1}}$, from which it follows that

$$F''(V) \leq -\frac{n \cdot Ric_0}{n-1}F(V)^{-\frac{n-2}{n}} \tag{3.9}$$

The correct definition for $m_{Ric}(V)$ then becomes

$$m_{Ric}(V) = \left(n^2(\omega_{n-1})^{\frac{2}{n-1}} - F'(V)^2\right) - \frac{n^2 \cdot Ric_0}{n-1}F(V)^{\frac{2}{n}} \tag{3.10}$$

where ω_{n-1} is the surface area of the sphere S^{n-1} of radius 1 in $\mathbf{R}^n$. As before, on the interval $[0, \frac{1}{2}\text{Vol}(M^n)]$, $F'(V) \geq 0$ and $m_{Ric}(V)$ is a nonnegative, nondecreasing function of V. The proof is the same as before.

Now consider phase space which we will view as the x-y plane where $x = F(V)$ and $y = F'(V)$. Let γ be the path in phase space of F(V) for V between 0 and $\frac{1}{2}\text{Vol}(M^n)$. Then we note that

$$\frac{1}{2}\text{Vol}(M^n) = \int_\gamma dV = \int_\gamma \frac{dx}{y} \tag{3.11}$$

We also observe that since $F(0) = 0$ and $F'(\frac{1}{2}\text{Vol}(M^n)) = 0$ (by the symmetry of $F(V)$), γ is a path from the y axis to the x axis. Since $F(V)$ is strictly increasing and $F'(V)$ is strictly decreasing (by inequality 3.9), the x position of γ is nondecreasing and the y position of γ is strictly decreasing. Since $F'(V)$ is sometimes multivalued, taking on the values of an interval, γ is sometimes vertical.

Now we want to find the γ which maximizes equation 3.11, with the constraint that $m_{Ric}(V)$ stays nonnegative and nondecreasing, which is equivalent to satisfying equation 3.9. Consider all the possible paths which terminate at a given point on the x axis, $(x_0, 0)$, and think of these paths as beginning at this point and then follow the paths backwards. The path which maximizes equation 3.11 will be the one which has the smallest y values. Since $F''(V) = y\frac{dy}{dx}$, we can rewrite inequality 3.9 as

$$\frac{dy}{dx} \leq -\frac{n \cdot Ric_0}{n-1} x^{-\frac{1}{3}} y^{-1} \tag{3.12}$$

The γ terminating at $(x_0, 0)$ with the smallest y values will be the path which has equality in inequality 3.12.

Hence, this path is given by the $F(V)$ which has equality in inequality 3.9. But equality for inequality 3.9 is equivalent to $m'_{Ric}(V) = 0$, which implies that $m_{Ric}(V) = m_0$, where m_0 is some positive constant. By equation 3.10, this path can be computed explicitly and is given by

$$m_0 = \left(n^2 (c_{n-1})^{\frac{2}{n-1}} - y^2\right) - \frac{n^2 \cdot Ric_0}{n-1} x^{\frac{2}{n}}$$

which can be rewritten as

$$y = \left[\left(n^2 (c_{n-1})^{\frac{2}{n-1}} - m_0\right) - \frac{n^2 \cdot Ric_0}{n-1} x^{\frac{2}{n}}\right]^{\frac{1}{2}} \tag{3.13}$$

Different values of m_0 correspond to curves terminating at different points on the x axis. Hence, the γ which maximizes equation 3.11 is a curve which is the graph of equation 3.13 for some m_0. By a simple change of variables, it is easy to compute that

$$\begin{aligned}
\frac{1}{2}\mathrm{Vol}(M^n) &= \int_\gamma \frac{dx}{y} \leq \sup_\gamma \int_\gamma \frac{dx}{y} \\
&= \sup_{m_0} \left(n^2 (c_{n-1})^{\frac{2}{n-1}} - m_0\right)^{\frac{n-1}{2}} \left(\frac{n-1}{n^2 \cdot Ric_0}\right)^{\frac{n}{2}} \int_0^1 \left[1 - z^{\frac{2}{n}}\right]^{-\frac{1}{2}} dz
\end{aligned}$$

Now we recall that $m(V)$ is nonnegative, so m_0 must also be nonnegative. Hence, the above expression is maximized when $m_0 = 0$. But the standard sphere (S^n, g_0) with constant Ricci curvature $Ric_0 \cdot g_0$ has $m_{Ric}(V) = 0$. This can be verified by direct computation using the fact that the isoperimetric spheres of (S^n, g_0) are the spherically symmetric $(n-1)$-spheres, or from noticing that since we get equality in inequality 3.8 when $(M^n, g) = (S^n, g_0)$, $m'(0) \equiv 0$, so $m(0) \equiv 0$. Let γ_0 be the path in phase space corresponding to this standard sphere with zero mass. Then

$$\frac{1}{2}\mathrm{Vol}(M^n) = \int_\gamma \frac{dx}{y} \leq \sup_\gamma \int_\gamma \frac{dx}{y} = \int_{\gamma_0} \frac{dx}{y} = \frac{1}{2}\mathrm{Vol}(S^n) \tag{3.14}$$

proving the theorem. □

3.4 Proof of the Volume Comparison Theorems involving Scalar Curvature

Proof. The approach we take here is the same as we used to prove Bishop's theorem in section 3.3. Going back to section 3.2 and combining equations 3.6 and 3.7, we get

$$F''(V) \leq \min\left\{\frac{36\pi - F'(V)^2}{6F(V)} - \frac{3R_0}{4}F(V)^{-\frac{1}{3}}, -\frac{3\epsilon \cdot Ric_0}{2}F(V)^{-\frac{1}{3}}\right\} \tag{3.15}$$

Since

$$m_{Ric}(V) = \left(36\pi - F'(V)^2\right) - \frac{9\epsilon \cdot Ric_0}{2}F(V)^{\frac{2}{3}} \tag{3.16}$$

and

$$m_R(V) = F(V)^{\frac{1}{3}}\left(36\pi - F'(V)^2\right) - \frac{3R_0}{2}F(V) \tag{3.17}$$

we can rewrite inequality 3.15 as

$$F''(V) \leq -\frac{1}{2}F^{-\frac{1}{3}} \cdot \max\left\{L(V), 3\epsilon \cdot Ric_0\right\} \tag{3.18}$$

where

$$L(V) = R_0 - \frac{m_R(V)}{3F} = \frac{3}{2}(R_0 - \epsilon \cdot Ric_0) - \frac{m_{Ric}(V)}{3F^{\frac{2}{3}}} \tag{3.19}$$

As before, we consider phase space which we will view as the x-y plane where $x = F(V)$ and $y = F'(V)$. Let γ be the path in phase space of $F(V)$ for V between

0 and $\frac{1}{2}\mathrm{Vol}(M^3)$. Then we recall that

$$\frac{1}{2}\mathrm{Vol}(M^3) = \int_\gamma dV = \int_\gamma \frac{dx}{y} \tag{3.20}$$

Since $F(0) = 0$ and $F'(\frac{1}{2}\mathrm{Vol}(M^3)) = 0$ (by the symmetry of $F(V)$), γ is a path from the y axis to the x axis. Since $F(V)$ is strictly increasing and $F'(V)$ is strictly decreasing (by inequality 3.18), the x position of γ is nondecreasing and the y position of γ is strictly decreasing. Again, since $F'(V)$ is sometimes multivalued, taking on the values of an interval, γ is sometimes vertical.

We want to find the γ which maximizes equation 3.20, while still satisfying inequality 3.18. Consider all the possible paths which terminate at a given point on the x axis, $(x_0, 0)$, and think of these paths as beginning at this point and then follow the paths backwards. The path which maximizes equation 3.20 will be the one which has the smallest y values. Since $F''(V) = y\frac{dy}{dx}$, we can rewrite inequality 3.18 as

$$\frac{dy}{dx} \leq -\frac{1}{2}x^{-\frac{1}{3}}y^{-1} \cdot \max\{L(V), 3\epsilon \cdot Ric_0\} \tag{3.21}$$

where we think of $L(V)$, $m_R(V)$, and $m_{Ric}(V)$ as functions of x and y instead of $F(V)$ and $F'(V)$. The γ terminating at $(x_0, 0)$ with the smallest y values will be the path which has equality in inequality 3.21, and thus has equality in inequality 3.15. Let's call this path $\gamma(x_0)$. Then we see that $\gamma(x_0)$ maximizes equation 3.20 among all paths which terminate at $(x_0, 0)$.

By the computations in the proof of lemma 6, equality in inequality 3.15 is equivalent to either $m'_R(V) = 0$ or $m'_{Ric}(V) = 0$ for each V. This, combined with equation 3.19 and the fact that $F(V)$ is a strictly increasing function of V, gives us that $L(V)$ is strictly increasing as a function of V for the path $\gamma(x_0)$. Furthermore,

$$\begin{aligned} L(V) \geq 3\epsilon \cdot Ric_0 \quad &\Rightarrow \quad m'_R(V) = 0 \\ L(V) \leq 3\epsilon \cdot Ric_0 \quad &\Rightarrow \quad m'_{Ric}(V) = 0 \end{aligned}$$

Hence, we see that there are three cases.

Case 1: If $L(V)$ is always less than $3\epsilon \cdot Ric_0$, then $\gamma(x_0)$ is the curve given by $m_{Ric}(V) = c$ for some constant $c \geq 0$ which depends on x_0.

Case 2: If $L(V)$ is initially smaller than $3\epsilon \cdot Ric_0$ but becomes larger than $3\epsilon \cdot Ric_0$ for $V \geq \tilde{V}$, then $\gamma(x_0)$ will be the union of two segments of curves, one given by $m_{Ric}(V) = c_2$ for $V \leq \tilde{V}$ and the other given by $m_R(V) = c_1$ for $V \geq \tilde{V}$, for three constants $c_1, c_2 \geq 0$ and $\tilde{V}$ which depend on x_0.

Case 3: If $L(V)$ is always greater than $3\epsilon \cdot Ric_0$, then $m_R(V) = c$ for some constant $c \geq 0$. By equation 3.19, though, we see that if this constant were positive, then $L(V)$ would approach $-\infty$ for small F, thus violating our assumption that $L(V) \geq 3\epsilon \cdot Ric_0$. Hence, $c = 0$, so $m_R(V) = 0$ for all V.

From these observations, we explicitly compute $\gamma(x_0)$. We spare the reader some of the routine details and summarize the results. For convenience, we define two special values for x_0. Let

$$x_S = \left(\frac{24\pi}{R_0}\right)^{\frac{3}{2}} \quad \text{and} \quad x_{FB} = \left(\frac{8\pi}{R_0 - 2\epsilon \cdot Ric_0}\right)^{\frac{3}{2}}$$

The subscripts stand for "sphere" and "football," since the standard 3-sphere produces the curve $\gamma(x_S)$ and the metric (which turns out to have two singularities) which produces $\gamma(x_{FB})$ looks like an axially symmetric football (with two pointy ends) when embedded in $\mathbf{R}^4$.

For $0 \leq x_0 \leq x_{FB}$, $\gamma(x_0)$ is the graph of the function

$$y = \left[\frac{9\epsilon \cdot Ric_0}{2}(x_0^{\frac{2}{3}} - x^{\frac{2}{3}})\right]^{\frac{1}{2}} \tag{3.22}$$

These are the curves from case 1.

For $x_{FB} \leq x_0 \leq x_S$, $\gamma(x_0)$ is the graph of the function

$$y = \begin{cases} \left(36\pi - c_2 - \frac{9\epsilon \cdot Ric_0}{2} x^{\frac{2}{3}}\right)^{\frac{1}{2}} & , 0 \leq x \leq x_1 \\ \left(36\pi - c_1 x^{-\frac{1}{3}} - \frac{3R_0}{2} x^{\frac{2}{3}}\right)^{\frac{1}{2}} & , x_1 \leq x \leq x_0 \end{cases} \tag{3.23}$$

where

$$c_1 = x_0^{\frac{1}{3}} \left(36\pi - \frac{3R_0}{2} x_0^{\frac{2}{3}} \right)$$
$$x_1 = \frac{c_1}{3(R_0 - 3\epsilon \cdot Ric_0)}$$
$$c_2 = \frac{3}{2} \left[3(R_0 - 3\epsilon \cdot Ric_0) c_1^2 \right]^{\frac{1}{3}}$$

These are the curves from case 2, and the constants c_1 and c_2 are the same constants that are mentioned in case 2. Case 3 is also included here, and occurs when $x_0 = x_S$, which implies that $c_1 = x_1 = 0$.

There are no paths which terminate at $(x_0, 0)$ for $x_0 > x_S$. This follows from the definition of $m_R(V)$ in equation 3.17 and the fact that $m_R(V) \geq 0$.

Now let's define

$$W(x_0) = \int_{\gamma(x_0)} dV = \int_{\gamma(x_0)} \frac{dx}{y} \tag{3.24}$$

Then we have that

$$\frac{1}{2}V = \frac{1}{2}\mathrm{Vol}(M^3) = \int_\gamma \frac{dx}{y} \leq \sup_\gamma \int_\gamma \frac{dx}{y} = \sup_{x_0} W(x_0) \tag{3.25}$$

where M^3 is any arbitrary 3-manifold satisfying the curvature conditions of theorem 19. Using equations 3.22 and 3.23, we can compute $W(x_0)$ explicitly.

$$W(x_0) = \begin{cases} \left(\frac{9\epsilon \cdot Ric_0}{2}\right)^{-\frac{1}{2}} x_0^{\frac{2}{3}} \int_0^1 \left(1 - z^{\frac{2}{3}}\right)^{-\frac{1}{2}} dz, & 0 \leq x_0 \leq x_{FB} \\ \\ \int_0^{x_1} \left(36\pi - c_2 - \frac{9\epsilon \cdot Ric_0}{2} x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx & \\ + \int_{x_1}^{x_0} \left(36\pi - c_1 x^{-\frac{1}{3}} - \frac{3R_0}{2} x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx, & x_{FB} < x_0 \leq x_S \end{cases} \tag{3.26}$$

where we've simplified the top integral using a change of variables. Unfortunately, it is not so easy to simplify the bottom integral. However, we can simplify the bottom integral when x_0 equals x_{FB} or x_S (because the values of c_1 and x_1 work out nicely), and we find that

$$W(x_S) = \frac{36\pi}{\left(\frac{9Ric_0}{2}\right)^{\frac{3}{2}}} \int_0^1 \left(1 - z^{\frac{2}{3}}\right)^{-\frac{1}{2}} dz \tag{3.27}$$

(using $R_0 = 3Ric_0$) and

$$W(x_{FB}) = \frac{x_{FB}^{\frac{2}{3}}}{\left(\frac{9\epsilon \cdot Ric_0}{2}\right)^{\frac{1}{2}}} \int_0^1 \left(1 - z^{\frac{2}{3}}\right)^{-\frac{1}{2}} dz \tag{3.28}$$

Now

$$\int_0^1 \left(1 - z^{\frac{2}{3}}\right)^{-\frac{1}{2}} dz = 3\pi/4 \tag{3.29}$$

so it is easy to check that

$$W(x_S) = \frac{1}{2}\mathrm{Vol}(S^3, g_0) = \frac{1}{2}V_0.$$

Furthermore, by the definition of x_{FB},

$$W(x_{FB}) = W(x_S)\frac{1}{\epsilon^{\frac{1}{2}}(3 - 2\epsilon)} \tag{3.30}$$

We can simplify things further if we recognize the fact that everything scales as it should. Using the values from the 3-sphere of radius one embedded in $\mathbf{R}^4$, we use $R_0 = 6$, $Ric_0 = 2$, and $V_0 = 2\pi^2$ to get

$$x_S = (4\pi)^{\frac{3}{2}}$$

$$x_{FB} = \left(\frac{4\pi}{3 - 2\epsilon}\right)^{\frac{3}{2}}$$

$$c_1 = x_0^{\frac{1}{3}}(36\pi - 9x_0^{\frac{2}{3}})$$

$$x_1 = \frac{c_1}{18(1 - \epsilon)}$$

$$c_2 = \frac{3}{2}[18(1 - \epsilon)c_1^2]^{\frac{1}{3}}$$

so that plugging in these values for W and scaling appropriately we get

$$V \leq \alpha(\epsilon) \cdot V_0$$

where

$$\alpha(\epsilon) = \sup_{0 \leq x_0 \leq (4\pi)^{\frac{3}{2}}} w_\epsilon(x_0)$$

where

$$w_\epsilon(x_0) = \frac{1}{\pi^2}\begin{cases} \frac{\pi}{4}\cdot\epsilon^{-\frac{1}{2}}\cdot x_0^{\frac{2}{3}}, & 0 \leq x_0 \leq \left(\frac{4\pi}{3-2\epsilon}\right)^{\frac{3}{2}} \\ \int_0^{\frac{c_1}{18(1-\epsilon)}} \left(36\pi - \frac{3}{2}[18(1-\epsilon)c_1^2]^{\frac{1}{3}} - 9\epsilon\cdot x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx & \\ \quad + \int_{\frac{c_1}{18(1-\epsilon)}}^{x_0} (36\pi - c_1 x^{-\frac{1}{3}} - 9x^{\frac{2}{3}})^{-\frac{1}{2}} dx, & \left(\frac{4\pi}{3-2\epsilon}\right)^{\frac{3}{2}} < x_0 \leq (4\pi)^{\frac{3}{2}} \end{cases}$$

where again

$$c_1 = x_0^{\frac{1}{3}}(36\pi - 9x_0^{\frac{2}{3}}).$$

We can simplify the notation a bit by changing variables. Let $x_0 = z^{\frac{3}{2}}$ and $c_1 = 18(1-\epsilon)y$. Then we have

$$\alpha(\epsilon) = \sup_{0 \leq z \leq 4\pi} w_\epsilon(z)$$

where

$$w_\epsilon(z) = \frac{1}{\pi^2}\begin{cases} \frac{\pi}{4}\cdot\epsilon^{-\frac{1}{2}}\cdot z, & 0 \leq z \leq \frac{4\pi}{3-2\epsilon} \\ \int_0^{y(z)} \left(36\pi - 27(1-\epsilon)y(z)^{\frac{2}{3}} - 9\epsilon\cdot x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx & \\ + \int_{y(z)}^{z^{\frac{3}{2}}} (36\pi - 18(1-\epsilon)y(z)x^{-\frac{1}{3}} - 9x^{\frac{2}{3}})^{-\frac{1}{2}} dx, & \frac{4\pi}{3-2\epsilon} < z \leq 4\pi \end{cases}$$

where

$$y(z) = \frac{z^{\frac{1}{2}}(4\pi - z)}{2(1-\epsilon)}.$$

Since w_ϵ is continuous, the maximum value must occur for $z \in [\frac{4\pi}{3-2\epsilon}, 4\pi]$. Hence, we have the following theorem.

Theorem 21 *Let (S^3, g_0) be the constant curvature metric on S^3 with scalar curvature R_0, Ricci curvature $Ric_0 \cdot g_0$, and volume V_0. If $\epsilon \in (0,1]$ and (M^3, g) is any complete smooth Riemannian manifold of volume V satisfying*

$$R(g) \geq R_0$$

$$Ric(g) \geq \epsilon \cdot Ric_0 \cdot g$$

then

$$V \leq \alpha(\epsilon) V_0$$

where

$$\alpha(\epsilon) = \sup_{\frac{4\pi}{3-2\epsilon} \leq z \leq 4\pi} \frac{1}{\pi^2} \left(\begin{array}{c} \int_0^{y(z)} \left(36\pi - 27(1-\epsilon) y(z)^{\frac{2}{3}} - 9\epsilon \cdot x^{\frac{2}{3}}\right)^{-\frac{1}{2}} dx \\ + \int_{y(z)}^{z^{\frac{3}{2}}} (36\pi - 18(1-\epsilon) y(z) x^{-\frac{1}{3}} - 9x^{\frac{2}{3}})^{-\frac{1}{2}} dx \end{array} \right)$$

where

$$y(z) = \frac{z^{\frac{1}{2}}(4\pi - z)}{2(1-\epsilon)}.$$

Furthermore, this expression for $\alpha(\epsilon)$ is sharp.

We note that the reason that this expression for $\alpha(\epsilon)$ is sharp is that it is possible to construct spherically symmetric manifolds which satisfy the curvature conditions of theorem 21 and have volumes as close to $\alpha(\epsilon)V_0$ as desired, and equal to $\alpha(\epsilon)V_0$ if we allow the manifolds to have singularities. These manifolds look like long and skinny axially symmetric footballs when embedded in $\mathbf{R}^4$ with two pointy ends where the manifold is not smooth. The smaller ϵ is, the longer and skinnier these manifolds become, and as ϵ goes to zero, these "case of equality" manifolds converge to the standard cylinder $S^2 \times \mathbf{R}$ which has constant scalar curvature R_0 (and zero Ricci curvature in the directions along the length of the cylinder). These manifolds can be constructed by looking at the function $W(x_0)$ from equation 3.26 (for each value of ϵ) and defining $\bar{x}(\epsilon)$ to be the value of x_0 which maximizes W. Then the curve $\gamma(\bar{x}(\epsilon))$ in phase space as described before corresponds to an $F(V)$ function, which yields an $A(V)$ function using $F(V) = A(V)^{3/2}$. Given an $A(V)$ function, we can then construct a spherically symmetric manifold such that the spherically symmetric spheres which contain a volume V have surface area $A(V)$, and it is easy to verify that these are in fact "case of equality" manifolds.

Direct computation shows that $w_\epsilon(z)$ is a C^1 function on $[0, 4\pi]$ and that $w_\epsilon(4\pi) = 1$, so $\alpha(\epsilon) \geq 1$ for all $\epsilon \in (0, 1]$. Furthermore, since direct calculation also shows that $w_\epsilon(z)$ is a nonincreasing function of ϵ when z is held fixed, it follows that $\alpha(\epsilon)$ is nonincreasing. Hence, if α equals one at one value of ϵ, then α equals one for all larger values of ϵ in the interval $(0, 1]$. Let

$$\epsilon_0 = \inf\{\epsilon \in (0, 1] \mid \alpha(\epsilon) = 1\}$$

Then we have the following theorem.

Theorem 22 *Let (S^3, g_0) be the constant curvature metric on S^3 with scalar curvature R_0, Ricci curvature $Ric_0 \cdot g_0$, and volume V_0. If (M^3, g) is any complete smooth Riemannian manifold of volume V satisfying*

$$R(g) \geq R_0$$

$$Ric(g) \geq \epsilon_0 \cdot Ric_0 \cdot g$$

then

$$V \leq V_0.$$

Naturally it would be desirable to estimate the actual value of ϵ_0. It is straightforward (although messy) to show that $\epsilon_0 < 1$. However, getting an accurate estimate for ϵ_0 definitely seems to be a job for a computer, and it seems reasonable to conjecture that ϵ_0 is transcendental. From preliminary computer calculations, it looks like $.134 < \epsilon_0 < .135$, although these bounds are not rigorous.

3.5 Estimates for ϵ_0

The results of this section are due primarily to Kevin Iga of Stanford University, who wrote several computer programs using the C programming language on a Sun SPARC station 20 computer to estimate the value of ϵ_0 from theorem 22. We found that $\epsilon_0 \approx 0.134727$. However, the only rigorous bounds that we have are

$$0.133974 < 1 - \frac{\sqrt{3}}{2} < \epsilon_0 < 1, \tag{3.31}$$

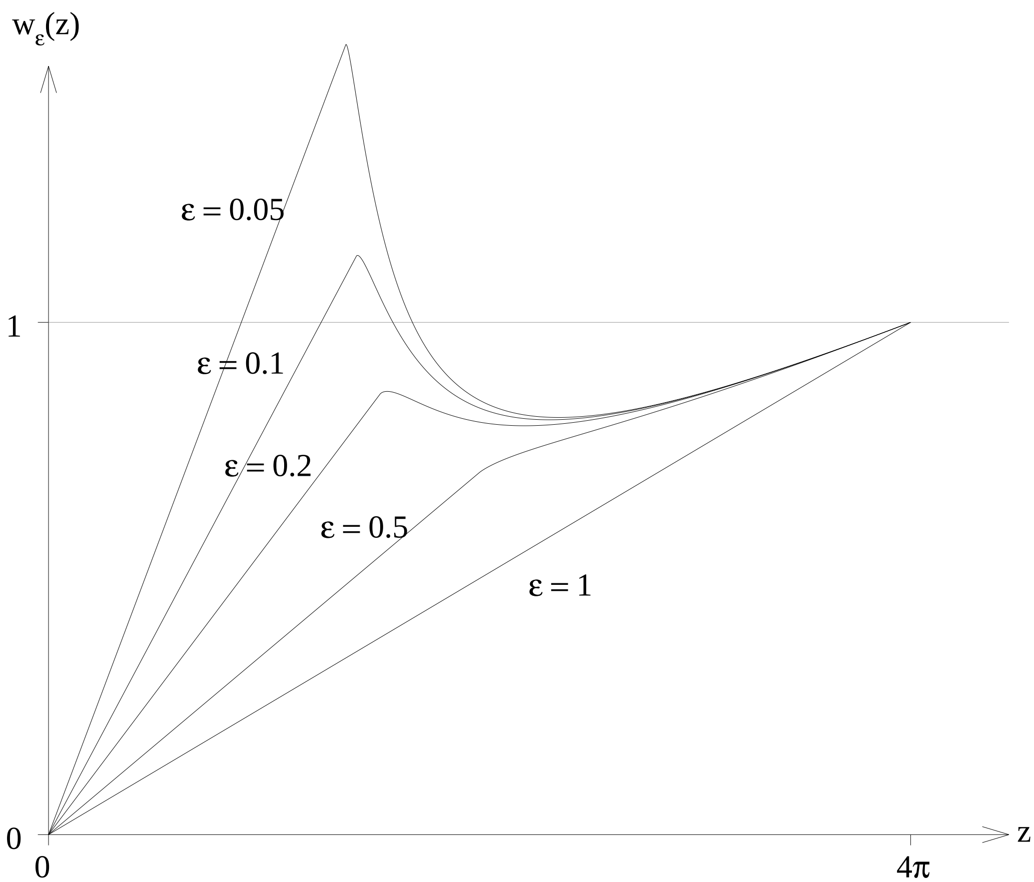

Figure 3.1: Graphs of $w_\epsilon(z)$ for $\epsilon = 0.05, 0.1, 0.2, 0.5$, and 1.

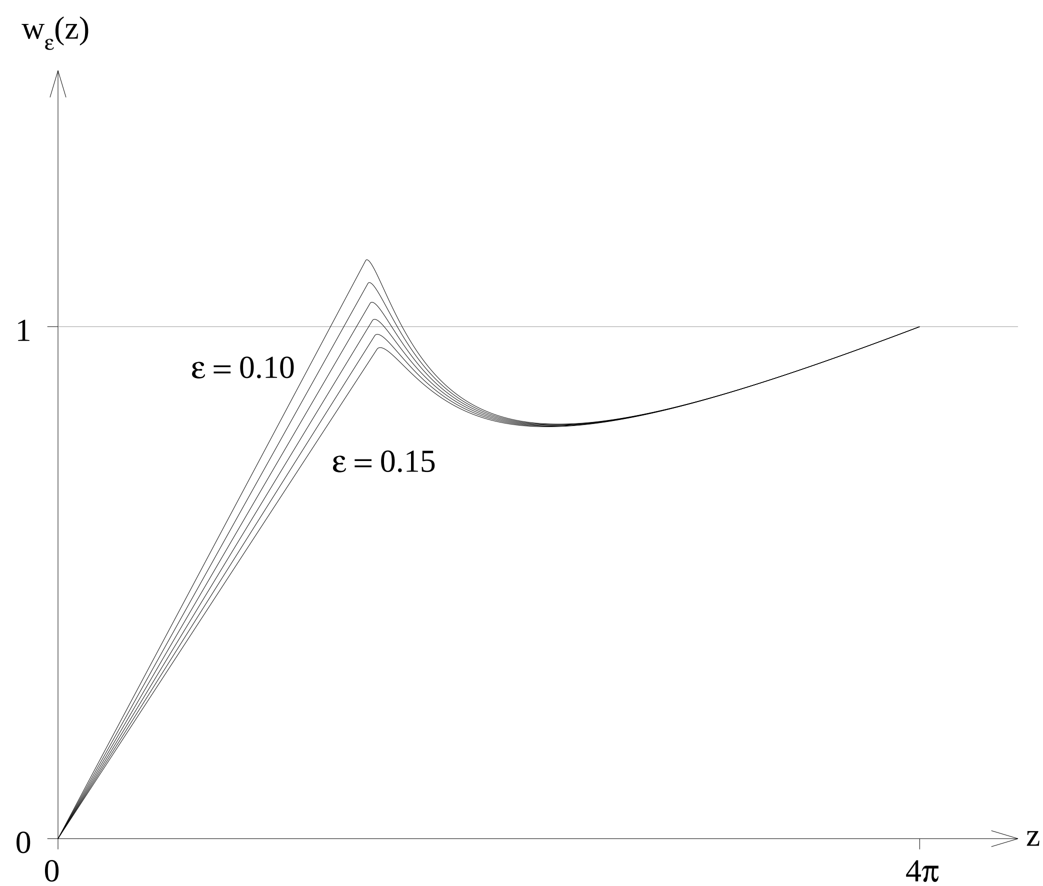

Figure 3.2: Graphs of $w_\epsilon(z)$ for $\epsilon = 0.10, 0.11, 0.12, 0.13, 0.14$, and 0.15.

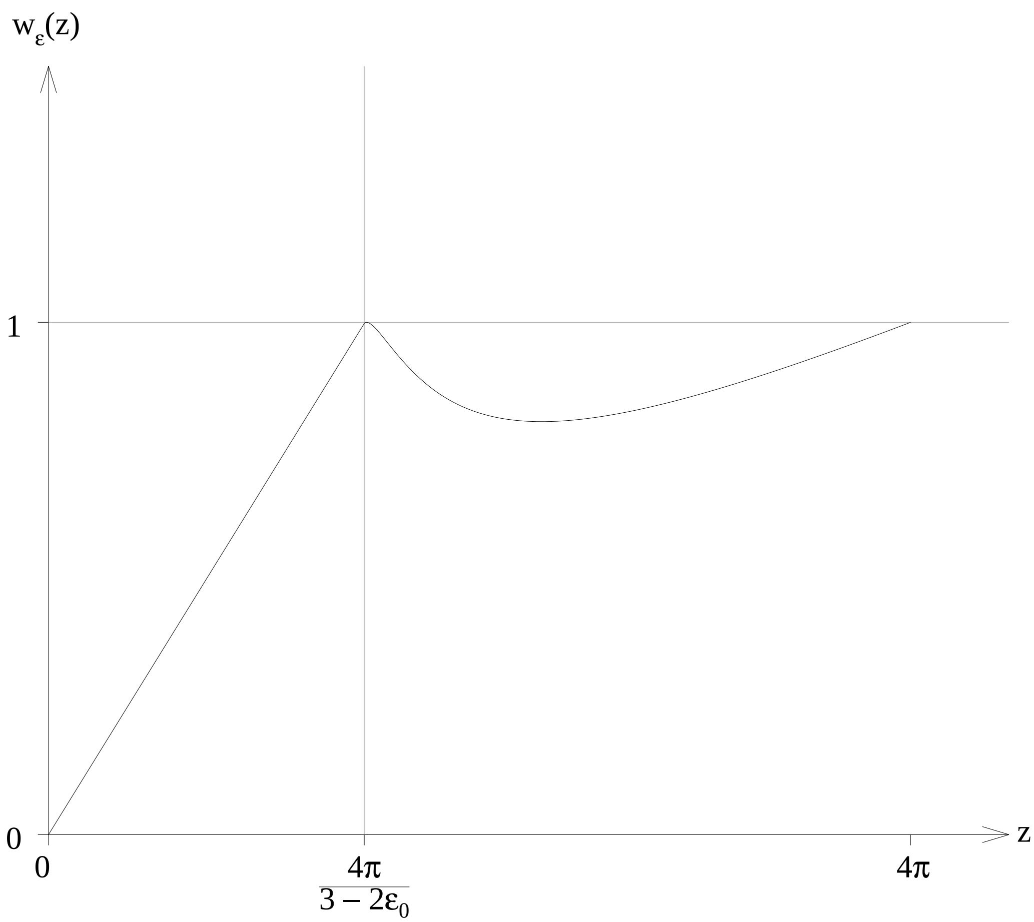

Figure 3.3: Graph of $w_\epsilon(z)$ for $\epsilon = 0.134727$.

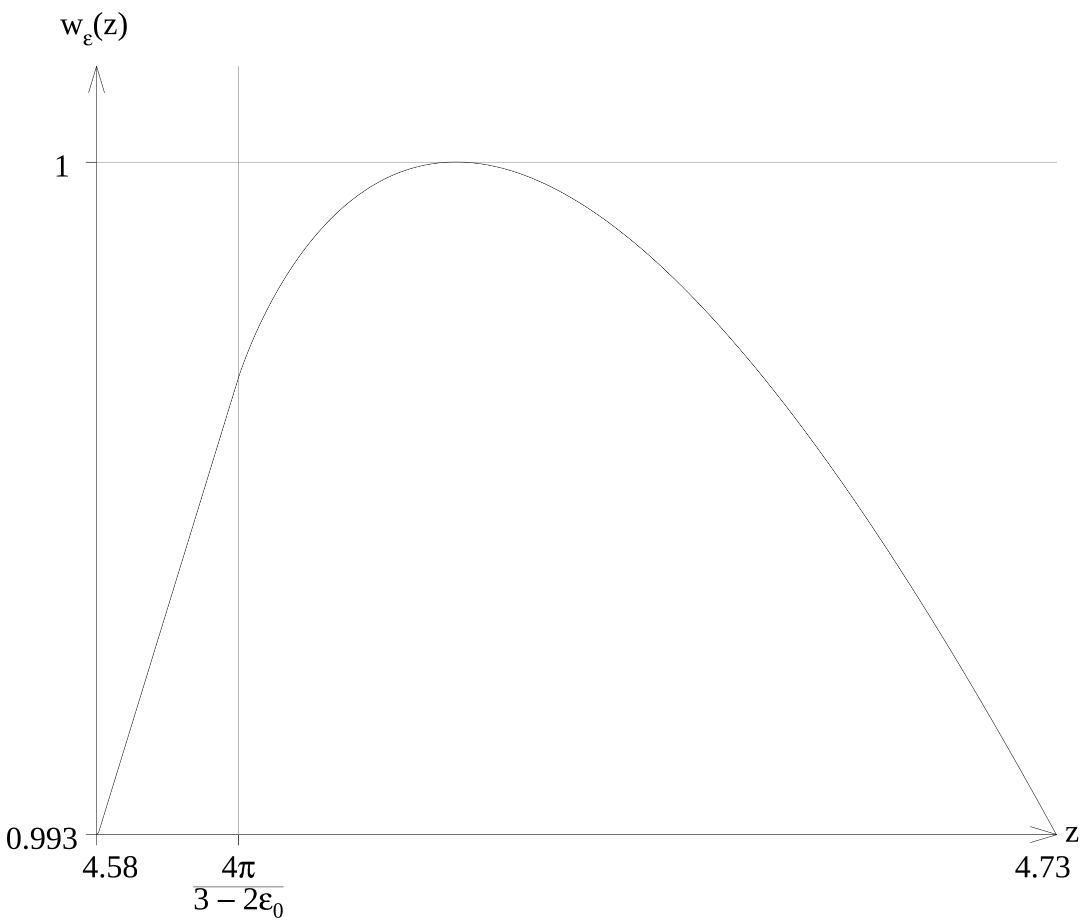

Figure 3.4: Graph of $w_\epsilon(z)$ for $\epsilon = 0.134727$, centered on the interior maximum point.

but we are reasonably confident that $0.134 < \epsilon_0 < 0.135$. We leave it those with greater expertise with computational methods to find better rigorous upper and lower bounds for ϵ_0.

We recall that

$$\epsilon_0 = \inf\{\epsilon \in (0,1] \mid \alpha(\epsilon) = 1\}$$

where

$$\alpha(\epsilon) = \sup_{0 \le z \le 4\pi} w_\epsilon(z)$$

where

$$w_\epsilon(z) = \frac{1}{\pi^2}\begin{cases} \frac{\pi}{4}\cdot\epsilon^{-\frac{1}{2}}\cdot z, & 0 \le z \le \frac{4\pi}{3-2\epsilon} \\ \int_0^{y(z)}\left(36\pi - 27(1-\epsilon)y(z)^{\frac{2}{3}} - 9\epsilon\cdot x^{\frac{2}{3}}\right)^{-\frac{1}{2}}dx & \\ +\int_{y(z)}^{z^{\frac{3}{2}}}(36\pi - 18(1-\epsilon)y(z)x^{-\frac{1}{3}} - 9x^{\frac{2}{3}})^{-\frac{1}{2}}dx, & \frac{4\pi}{3-2\epsilon} < z \le 4\pi \end{cases}$$

where

$$y(z) = \frac{z^{\frac{1}{2}}(4\pi - z)}{2(1-\epsilon)}.$$

We recall that since $w_\epsilon(z)$ is continuous, the maximum value must occur for $z \in [\frac{4\pi}{3-2\epsilon}, 4\pi]$. In fact, since $w_\epsilon(z)$ is C^1, the maximum value can not occur on the left end point of this interval, $z = \frac{4\pi}{3-2\epsilon}$, although there is a local maximum very close to this point when ϵ is less than about 0.2. In fact, we find that the maximum value of $w_\epsilon(z)$ either occurs at $z = 1$ or at a z value only slightly greater than $\frac{4\pi}{3-2\epsilon}$. This phenomenon can be seen in figure 3.1 where we can recognize the location of $z = \frac{4\pi}{3-2\epsilon}$ on the graph using the fact that $w_\epsilon(z)$ is linear for $0 \le z \le \frac{4\pi}{3-2\epsilon}$.

As previously mentioned, it is easily shown that $w_\epsilon(z)$ is a decreasing function of ϵ when z is held fixed. Thus, from figure 3.1 we see that the maximum value of $w_\epsilon(z)$ is greater than one when $\epsilon = 0.1$, so this must be the case for all $\epsilon < 0.1$ as well. Hence, $\epsilon_0 > 0.1$. Using this idea again we conclude from figure 3.2 that $0.13 < \epsilon_0 < 0.14$, and continuing this procedure is how we estimated that $\epsilon_0 \approx 0.134727$.

In figure 3.3 we see the graph of $w_\epsilon(z)$ when $\epsilon = 0.134727$, so that the maximum value of $w_\epsilon(z)$ is roughly one and occurs (to the accuracy of the computer) at two z values, $z = 1$ and z slightly greater than $\frac{4\pi}{3-2\epsilon}$. Figure 3.4 is an enlargement of figure

3.3 around this second maximum.

Notice that from the form of the formula for $w_\epsilon(z)$ that solving for the explicit values of the critical points using $w_\epsilon'(z) = 0$ seems very difficult, and this is why it seems necessary to resort to numerical computations.

The first integral in the formula for $w_\epsilon(z)$ can be computed in closed form. However, we used Simpson's rule to estimate the integral in the formula for $w_\epsilon(z)$. To use Simpson's rule, we need the function we are integrating to be bounded, so we subtract the function $k(1 - \frac{x}{z^{3/2}})^{-1/2}$ for some k from the second integrand to make it a bounded function. We then use Simpson's rule with N, the number of intervals, equal to one thousand. We have not attempted any rigorous error estimates, although we have observed that the value of ϵ_0 which we compute is the same to six digits for $N = 100$, which is a good sign.

To get the rigorous bounds in inequality 3.31, we note that $w_\epsilon(\frac{4\pi}{3-2\epsilon}) = \frac{1}{\epsilon^{\frac{1}{2}}(3-2\epsilon)} = 1$ when $\epsilon = 1 - \sqrt{3}/2$. Since this endpoint is never the maximum value of $w_\epsilon(z)$ since $w_\epsilon(z)$ is C^1 and has positive slope at $z = \frac{4\pi}{3-2\epsilon}$, $\alpha(1-\sqrt{3}/2) > 1$. Hence, $\epsilon_0 > 1-\sqrt{3}/2$. Finally, to show that $\epsilon_0 < 1$, it is sufficient to prove that for some $\epsilon < 1$, $w_\epsilon'(z) \geq 0$ which implies that the maximum value of $w_\epsilon(z)$ occurs at $z = 4\pi$ and equals 1. Choosing ϵ very close to 1 we find that this is true, although the computations are not trivial. Thus, $\alpha(\epsilon) = 1$ for some $\epsilon < 1$, so $\epsilon_0 < 1$.

3.6 Conjectures

The most natural generalization of theorem 22 is to propose that it is true in higher dimensions.

Conjecture 4 *Let (S^n, g_0) be the constant curvature metric on S^n with scalar curvature R_0, Ricci curvature $Ric_0 \cdot g_0$, and volume V_0. Then for each $n \geq 3$, there exists a positive $\epsilon_0(n) < 1$ such that if (M^n, g) is any complete smooth Riemannian manifold with volume V satisfying*

$$R(g) \geq R_0$$

$$Ric(g) \geq \epsilon_0(n) \cdot Ric_0 \cdot g$$

then

$$V \leq V_0.$$

Other problems which relate to scalar curvature include questions connected to the Yamabe problem and Einstein metrics [25]. Given a manifold M^n, consider metrics of volume one and define the energy to be the integral of scalar curvature. Einstein metrics are critical points of this functional. One approach to finding critical points of energy is to define $I(g)$ to be the infimum of the energy of all metrics conformal to g, and then to define $\sigma(M)$ to be the supremum of $I(g)$ over all conformal classes of metrics. If $\sigma(M) \leq 0$, then it is known that $I(g)$ is always realized by a unique metric, so that $\sigma(M)$ is realized by a metric which is a critical point of the energy functional and hence is Einstein. However, for $\sigma(M) > 0$, it is not known under what circumstances this procedure yields a critical point of the energy functional.

Also, if M is a manifold which admits a constant curvature metric, then it is conjectured by Schoen [25] that the above procedure produces the constant curvature metric and that $\sigma(M)$ equals the energy of this metric. Schoen's conjecture splits naturally into two cases, depending on whether the constant curvature metric is negatively curved or positively curved. Considering these two cases separately motivates the following two conjectures.

Conjecture 5 (Schoen) *Suppose M^n, $n \geq 2$, admits a hyperbolic metric g_0 with constant negative scalar curvature R_0. If g is any other metric on M^n with $R(g) \geq R_0$, then $Vol(g) \geq Vol(g_0)$.*

Conjecture 6 *Let (S^n, g_0), $n \geq 2$, be the standard constant curvature metric on S^n with first nonzero eigenvalue of the Laplacian operator λ_0. Let G be any finite isometric group action on (S^n, g_0) without fixed points, so that $(M^n, g_0) = (S^n, g_0)/G$ is a constant curvature metric on M^n with scalar curvature R_0 and volume V_0. If g is a metric on M^n with $R(g) \geq R_0$ and first eigenvalue $\lambda(g) \geq \lambda_0$, then $Vol(g) \leq V_0$.*

Conjectures 5 and 6 imply Schoen's conjecture respectively in the negatively and positively curved cases. (In the case that M^n admits a flat metric the conjecture is already known to be true.) Furthermore, if either conjecture 5 or 6 turns out to be false, then there would be a good chance that a counterexample to Schoen's conjecture could be found.

Conjectures 4 and 6 have the similarity that both attempt to use a lower bound on scalar curvature to achieve an upper bound on the total volume. However, both conjectures are false without additional assumptions. For conjecture 4, we need a lower bound on the Ricci curvature, and for conjecture 6 we need a lower bound on the first nonzero eigenvalue. These last two inequalities are weak in the sense that they are not equalities for the constant curvature metrics (unless G is trivial in conjecture 6). Hence, both conjectures say that for metrics close to the constant curvature metric (on S^n in conjecture 4 and on S^n/G in conjecture 6, for nontrivial G) that $R \geq R_0$ implies $V \leq V_0$. Conjecture 5, on the other hand, is a volume comparison conjecture for scalar curvature for hyperbolic metrics, and is particularly compelling because of its simplicity.

Appendix A

Some Geometric Calculations

Let Σ^2 be a smooth compact surface without boundary in (M^3, g). In this appendix we compute the rate of change of the mean curvature and the area form of Σ^2 given a smooth variation of Σ^2. We define a variation of Σ^2 as follows. For $-\epsilon < t < \epsilon$ and $x \in \Sigma^2$, suppose $\Sigma^2(x,t)$ takes values in M^3, is smooth, $\Sigma^2(t) = \{\Sigma^2(x,t) | x \in \Sigma^2\}$ is a smooth family of surfaces around Σ^2, and the vector $\frac{\partial \Sigma^2(x,t)}{\partial t}$ is perpendicular to $\Sigma^2(t)$ at $\Sigma^2(x,t)$. Let $\vec{\mu}(x,t)$ be the outward-pointing unit normal to $\Sigma^2(t)$ at $\Sigma^2(x,t)$, so that we must have

$$\frac{\partial \Sigma^2(x,t)}{\partial t} = \eta(x,t)\vec{\mu}(x,t) \tag{A.1}$$

for some real-valued function $\eta(x,t)$. Then we see that the surfaces $\Sigma^2(t_0)$ can be thought of as the surface created by starting at Σ^2 and flowing in the outward unit normal (to $\Sigma^2(t)$) direction at speed $\eta(x,t)$ for t between 0 and t_0. We call $\eta(x,t)$ the flow rate. In fact, given any smooth flow rate $\eta(x,t)$, for $x \in \Sigma^2$ and $t \in (-\delta, \delta)$, we can always find a smooth mapping $\Sigma^2(x,t)$ as above satisfying equation (A.1) such that $\Sigma^2(t)$ is a smooth family of surfaces around Σ^2, for $t \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$.

Let $du(x)$ be the area form on Σ^2, $\pi(x)$ be the second fundamental form of Σ^2 in M^3 at x, and $H(x) = \mathrm{trace}(\pi(x))$ be the mean curvature of Σ^2 at x. Let $d\mu(x,t)$ be the area form on $\Sigma^2(t)$, $\pi(x,t)$ be the second fundamental form of $\Sigma^2(t)$ in M^3 at $\Sigma^2(x,t)$, and $H(x,t) = \mathrm{trace}(\pi(x,t))$ be the mean curvature of $\Sigma^2(t)$ at the point

$\Sigma^2(x,t)$, for $t \in (-\epsilon,\epsilon)$. In this section we will verify the formulas

$$\frac{\partial}{\partial t} d\mu(x,t) = H(x,t)\eta(x,t)d\mu(x,t) \tag{A.2}$$

and

$$\frac{\partial}{\partial t} H(x,t) = -\Delta_{\Sigma(t)}\eta(x,t) - \eta(x,t)\|\pi(x,t)\|^2_{M^3} - \eta(x,t)Ric_{M^3}(\vec{\mu}(x,t),\vec{\mu}(x,t)) \tag{A.3}$$

which we will use for important calculations in chapters 2 and 3.

Let $\alpha : U \to \Sigma^2$ for some $U \subset \mathbf{R}^2$ be a local coordinate chart for Σ^2. Then we can define $\Sigma^2(x,t)$ equivalently locally on $U \times [-\epsilon,\epsilon] \subset \mathbf{R}^3$ with coordinates (x_1,x_2,t). Let ∂_i be the vector $\frac{\partial}{\partial x_i}$, and define the 2×2 matrix

$$g_{ij}(x_1,x_2,t) = \langle \partial_i, \partial_j \rangle_{M^3} \ , \ \ 1 \le i,j \le 2$$

where $\langle \cdot,\cdot \rangle_{M^3}$ is the pull-back of the metric of M^3 using the mapping $\Sigma^2(x,t)$: $U \times [-\epsilon,\epsilon] \to M^3$. Then $g_{ij}(x_1,x_2,t)$ is the metric for some neighborhood of $\Sigma^2(t)$ so that

$$d\mu(x_1,x_2,t) = \sqrt{|g(x_1,x_2,t)|}\, dx_1 \wedge dx_2$$

where $|g(x_1,x_2,t)| = \det(\{g_{ij}(x_1,x_2,t)\})$. Computing, we get

$$\frac{\partial}{\partial t}\sqrt{|g|} = \frac{1}{2}|g|^{-1/2}\frac{\partial}{\partial t}|g| = \frac{1}{2}|g|^{1/2}\mathrm{trace}(g^{-1}\frac{\partial}{\partial t}g)$$

where we have used the formula $\frac{\partial}{\partial t}(\det A) = (\det A)\mathrm{trace}(A^{-1}\frac{\partial}{\partial t}A)$. Also,

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\langle \partial_i, \partial_j \rangle_{M^3} \\ &= \langle D_{\partial_t}\partial_i, \partial_j \rangle_{M^3} + \langle \partial_i, D_{\partial_t}\partial_j \rangle_{M^3} \\ &= \langle D_{\partial_i}\partial_t, \partial_j \rangle_{M^3} + \langle \partial_i, D_{\partial_j}\partial_t \rangle_{M^3} \end{aligned}$$

since $D_{\partial_i}\partial_t - D_{\partial_t}\partial_i = [\partial_i, \partial_t] = 0$ by the torsion-free property of the connection in M^3

and since ∂_i and ∂_t are coordinate vectors. Thus, since by equation (A.1) $\partial_t = \eta\vec{\mu}$,

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} &= \langle D_{\partial_i}(\eta\vec{\mu}), \partial_j\rangle_{M^3} + \langle \partial_i, D_{\partial_j}(\eta\vec{\mu})\rangle_{M^3} \\ &= \eta\langle D_{\partial_i}\vec{\mu}, \partial_j\rangle_{M^3} + \eta\langle \partial_i, D_{\partial_j}\vec{\mu}\rangle_{M^3}\end{aligned}$$

since $\langle \vec{\mu}, \partial_i\rangle_{M^3} = 0$. Furthermore, since the second fundamental form is given by $\pi_{ij} = \langle D_{\partial_i}\vec{\mu}, \partial_j\rangle$ and is symmetric, we have

$$\frac{\partial}{\partial t} g_{ij} = 2\eta\pi_{ij}. \tag{A.4}$$

Thus, putting it all together, we have

$$\begin{aligned}\frac{\partial}{\partial t} d\mu(x_1, x_2, t) &= \frac{\partial}{\partial t}\sqrt{|g|}\, dx_1 \wedge dx_2 \\ &= \frac{1}{2}|g|^{1/2}\mathrm{trace}(g^{-1}\frac{\partial}{\partial t} g)\, dx_1 \wedge dx_2 \\ &= \frac{1}{2}|g|^{1/2}\mathrm{trace}(g^{ij} 2\eta\pi_{jk})\, dx_1 \wedge dx_2 \\ &= \mathrm{trace}(g^{ij}\pi_{jk})\eta\sqrt{|g|}\, dx_1 \wedge dx_2 \\ &= H(x_1, x_2, t)\eta(x_1, x_2, t)\, d\mu(x_1, x_2, t).\end{aligned}$$

Thus, equation (A.2) is true.

Now we verify equation (A.3). Since $H = g^{ij}\pi_{ij}$,

$$\frac{\partial}{\partial t} H = (\frac{\partial}{\partial t} g^{ij})\pi_{ij} + g^{ij}(\frac{\partial}{\partial t}\pi_{ij}).$$

But since $\frac{\partial}{\partial t}(A \cdot A^{-1}) = 0$, by the product rule it follows that $\frac{\partial}{\partial t}(A^{-1}) = -A^{-1}(\frac{\partial}{\partial t} A)A^{-1}$ so that by equation (A.4),

$$\begin{aligned}\frac{\partial}{\partial t} H &= g^{ij}(\frac{\partial}{\partial t}\pi_{ij}) - g^{ij} \cdot 2\eta\pi_{jk} \cdot g^{kl} \cdot \pi_{li} \\ &= g^{ij}(\frac{\partial}{\partial t}\pi_{ij}) - 2\eta \cdot \pi^i_k \pi^k_i \\ &= g^{ij}(\frac{\partial}{\partial t}\pi_{ij}) - 2\eta\|\pi\|^2_{M^3}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{\partial}{\partial t}\pi_{ij} &= \frac{\partial}{\partial t}\langle D_{\partial_i}\vec{\mu}, \partial_j\rangle \\ &= \langle D_{\partial_t}D_{\partial_i}\vec{\mu}, \partial_j\rangle + \langle D_{\partial_i}\vec{\mu}, D_{\partial_t}\partial_j\rangle \\ &= -R(\partial_t, \partial_i, \vec{\mu}, \partial_j) + \langle D_{\partial_i}D_{\partial_t}\vec{\mu}, \partial_j\rangle + \langle D_{\partial_i}\vec{\mu}, D_{\partial_t}\partial_j\rangle\end{aligned}$$

by the definition of the Riemann curvature tensor. Then since $D_{\partial_t}\partial_j = D_{\partial_j}\partial_t$ and $\partial_t = \eta\vec{\mu}$,

$$\frac{\partial}{\partial t}\pi_{ij} = -\eta \cdot R(\vec{\mu}, \partial_i, \vec{\mu}, \partial_j) + \langle D_{\partial_i}D_{\partial_t}\vec{\mu}, \partial_j\rangle + \langle D_{\partial_i}\vec{\mu}, D_{\partial_j}\partial_t\rangle.$$

We leave it to the reader to check that $D_{\partial_t}\vec{\mu} = -\vec{\nabla}_{\Sigma(t)}\eta$. Furthermore, since $\partial_t = \eta\vec{\mu}$, and $\langle D_{\partial_j}\vec{\mu}, \vec{\mu}\rangle = 0$,

$$\frac{\partial}{\partial t}\pi_{ij} = -\eta \cdot R(\vec{\mu}, \partial_i, \vec{\mu}, \partial_j) + \langle D_{\partial_i}(-\vec{\nabla}_{\Sigma(t)}\eta), \partial_j\rangle + \eta\langle D_{\partial_i}\vec{\mu}, D_{\partial_j}\vec{\mu}\rangle$$

so that

$$g^{ij}\frac{\partial}{\partial t}\pi_{ij} = -\eta Ric(\vec{\mu}, \vec{\mu}) - \Delta_{\Sigma(t)}\eta + \eta\|\pi\|^2_{M^3}$$

where $Ric(\cdot, \cdot)$ is the Ricci curvature tensor.

Hence, from before, we have

$$\frac{\partial}{\partial t}H = -\Delta_{\Sigma(t)}\eta - \eta\|\pi\|^2_{M^3} - \eta Ric(\mu, \mu)$$

proving equation (A.3).

One immediate consequence to equation (A.2) is that smooth surfaces which minimize area with a volume constraint must have constant mean curvature. Otherwise, we consider a flow on the surface Σ with a flow rate η defined on Σ. Then since the area of $\Sigma(t)$ is

$$A(t) = \int_{\Sigma(t)} d\mu(x, t)$$

we have that

$$A'(0) = \int_\Sigma \frac{\partial}{\partial t} d\mu(x,0) = \int_\Sigma H(x,0)\eta(x,0)\, d\mu(x,0).$$

Furthermore, since

$$V'(0) = \int_\Sigma \eta(x,0)$$

we can find an $\eta(x,0)$ such that $A'(0) < 0$ and $V'(0) = 0$ unless $H(x,0)$ equals a constant. Hence, any smooth surface which even locally minimizes area among surfaces containing the same volume must have constant mean curvature.

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