1 (a) Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + x^2 y' + xy = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} n c_n x^{n+1}}_{k=n+1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty}(k-1)c_{k-1}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k$$

$$= 2c_2 + (6c_3 + c_0)x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + kc_{k-1}]x^k = 0.$$

Thus

$$c_2 = 0$$

$$6c_3 + c_0 = 0$$

$$(k+2)(k+1)c_{k+2} + kc_{k-1} = 0$$

and

$$c_2 = 0$$

$$c_3 = -\frac{1}{6}c_0$$

$$c_{k+2} = -\frac{k}{(k+2)(k+1)}\,c_{k-1}, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_3 = -\frac{1}{6}$$

$$c_4 = c_5 = 0$$

$$c_6 = \frac{1}{45}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = 0$$

$$c_4 = -\frac{1}{6}$$

$$c_5 = c_6 = 0$$

$$c_7 = \frac{5}{252}$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{45}x^6 - \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^4 + \frac{5}{252}x^7 - \cdots .$$

1 (b) Substituting $y=\sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\left(x^2+1\right)y''-6y=\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n}+\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2}-6\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$=\sum_{k=2}^{\infty} k(k-1)c_k x^k+\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-6\sum_{k=0}^{\infty} c_k x^k$$

$$=2c_2-6c_0+(6c_3-6c_1)x+\sum_{k=2}^{\infty}\left[\left(k^2-k-6\right)c_k+(k+2)(k+1)c_{k+2}\right]x^k=0.$$

Thus

$$2c_2-6c_0=0$$

$$6c_3-6c_1=0$$

$$(k-3)(k+2)c_k+(k+2)(k+1)c_{k+2}=0$$

and

$$c_2=3c_0$$

$$c_3=c_1$$

$$c_{k+2}=-\frac{k-3}{k+1}c_k, \quad k=2,3,4,\ldots.$$

Choosing $c_0=1$ and $c_1=0$ we find

$$c_2=3$$

$$c_3=c_5=c_7=\cdots=0$$

$$c_4=1$$

$$c_6=-\frac{1}{5}$$

and so on. For $c_0=0$ and $c_1=1$ we obtain

$$c_2=c_4=c_6=\cdots=0$$

$$c_3=1$$

$$c_5=c_7=c_9=\cdots=0.$$

Thus, two solutions are

$$y_1=1+3x^2+x^4-\frac{1}{5}x^6+\cdots \qquad \text{and} \qquad y_2=x+x^3.$$

2 (a) Regular singular points: $x = 0,\ -3$

(b) Regular singular points: $x = 0,\ \pm 2i$

(c) Regular singular points: $x = -3,\ 2$

3 (a) Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$x^2 y'' - \left(x - \frac{2}{9}\right) y = \left(r^2 - r + \frac{2}{9}\right) c_0 x^r + \sum_{k=1}^{\infty} \left[(k+r)(k+r-1)c_k + \frac{2}{9}c_k - c_{k-1}\right] x^{k+r}$$
$$= 0,$$

which implies

$$r^2 - r + \frac{2}{9} = \left(r - \frac{2}{3}\right)\left(r - \frac{1}{3}\right) = 0$$

and

$$\left[(k+r)(k+r-1) + \frac{2}{9}\right] c_k - c_{k-1} = 0.$$

The indicial roots are $r = 2/3$ and $r = 1/3$. For $r = 2/3$ the recurrence relation is

$$c_k = \frac{3c_{k-1}}{3k^2 + k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{3}{4}c_0, \qquad c_2 = \frac{9}{56}c_0, \qquad c_3 = \frac{9}{560}c_0,$$

and so on. For $r = 1/3$ the recurrence relation is

$$c_k = \frac{3c_{k-1}}{3k^2 - k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{3}{2}c_0, \qquad c_2 = \frac{9}{20}c_0, \qquad c_3 = \frac{9}{160}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right) + C_2 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right).$$

3 (b) Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$2xy'' - (3+2x)y' + y = \left(2r^2 - 5r\right) c_0 x^{r-1} + \sum_{k=1}^{\infty} [2(k+r)(k+r-1)c_k - 3(k+r)c_k - 2(k+r-1)c_{k-1} + c_{k-1}]x^{k+r-1} = 0,$$

which implies

$$2r^2 - 5r = r(2r-5) = 0$$

and

$$(k+r)(2k+2r-5)c_k - (2k+2r-3)c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 5/2$. For $r = 0$ the recurrence relation is

$$c_k = \frac{(2k-3)c_{k-1}}{k(2k-5)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{1}{3}c_0, \qquad c_2 = -\frac{1}{6}c_0, \qquad c_3 = -\frac{1}{6}c_0,$$

and so on. For $r = 5/2$ the recurrence relation is

$$c_k = \frac{2(k+1)c_{k-1}}{k(2k+5)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{4}{7}c_0, \qquad c_2 = \frac{4}{21}c_0, \qquad c_3 = \frac{32}{693}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 + \cdots\right) + C_2 x^{5/2}\left(1 + \frac{4}{7}x + \frac{4}{21}x^2 + \frac{32}{693}x^3 + \cdots\right).$$

3 (c) Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$9x^2y'' + 9x^2y' + 2y = \left(9r^2 - 9r + 2\right) c_0 x^r + \sum_{k=1}^{\infty} [9(k+r)(k+r-1)c_k + 2c_k + 9(k+r-1)c_{k-1}]x^{k+r} = 0,$$

which implies

$$9r^2 - 9r + 2 = (3r-1)(3r-2) = 0$$

and

$$[9(k+r)(k+r-1) + 2]c_k + 9(k+r-1)c_{k-1} = 0.$$

The indicial roots are $r = 1/3$ and $r = 2/3$. For $r = 1/3$ the recurrence relation is

$$c_k = -\frac{(3k-2)c_{k-1}}{k(3k-1)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{1}{2}c_0, \qquad c_2 = \frac{1}{5}c_0, \qquad c_3 = -\frac{7}{120}c_0,$$

and so on. For $r = 2/3$ the recurrence relation is

$$c_k = -\frac{(3k-1)c_{k-1}}{k(3k+1)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{1}{2}c_0, \qquad c_2 = \frac{5}{28}c_0, \qquad c_3 = -\frac{1}{21}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 - \frac{7}{120}x^3 + \cdots\right) + C_2 x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 - \frac{1}{21}x^3 + \cdots\right).$$

4. Solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 17 = 0$$

we obtain eigenvalues $\lambda_1 = 1+4i$ and $\lambda_2 = 1-4i$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 4t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin 4t\right]e^t + c_2\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin 4t\right]e^t$$

$$= c_1\begin{pmatrix} -\sin 4t \\ \cos 4t \end{pmatrix}e^t + c_2\begin{pmatrix} -\cos 4t \\ -\sin 4t \end{pmatrix}e^t.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}t + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}e^{6t}$$

into the system yields

$$\begin{aligned} a_3 - 4b_3 &= -4 & a_2 - 4b_2 &= a_3 & -5a_1 - 4b_1 &= -9 \\ 4a_3 + b_3 &= 1 & 4a_2 + b_2 &= b_3 & 4a_1 - 5b_1 &= -1 \end{aligned}$$

from which we obtain $a_3 = 0$, $b_3 = 1$, $a_2 = 4/17$, $b_2 = 1/17$, $a_1 = 1$, and $b_1 = 1$. Then

$$\mathbf{X}(t) = c_1\begin{pmatrix} -\sin 4t \\ \cos 4t \end{pmatrix}e^t + c_2\begin{pmatrix} -\cos 4t \\ -\sin 4t \end{pmatrix}e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}t + \begin{pmatrix} 4/17 \\ 1/17 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6t}.$$

5. From

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix}e^{2t} + c_2\begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix}e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} -e^{2t}\sin 2t & e^{2t}\cos 2t \\ 2e^{2t}\cos 2t & 2e^{2t}\sin 2t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\frac{1}{2}e^{-2t}\sin 2t & \frac{1}{4}e^{-2t}\cos 2t \\ \frac{1}{2}e^{-2t}\cos 2t & \frac{1}{4}e^{-2t}\sin 2t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{1}{2}\cos 4t \\ \frac{1}{2}\sin 4t \end{pmatrix}dt = \begin{pmatrix} \frac{1}{8}\sin 4t \\ -\frac{1}{8}\cos 4t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -\frac{1}{8}\sin 2t\cos 4t - \frac{1}{8}\cos 2t\cos 4t \\ \frac{1}{4}\cos 2t\sin 4t - \frac{1}{4}\sin 2t\cos 4t \end{pmatrix}e^{2t}.$$