I tried to give a clue in my two posts as to how to see this in an elementary way, but it seems to have not been helpful. One really does need to carry out the experiments oneself to see it. At least I did. The one I described with a triangle at the corner of a cube helped me but here is another try.

Parallel transport measures exactly the same thing as the angle sum, or angle sum exess, of a geodesic triangle, or geodesic polygons (yes one can manage without geodesics, but it is easier to see what happens with them). Namely what is being measured is average curvature within the polygon. In Riemann's habilitationshrift he remarks that curvature at a point is angle sum excess or defect, compared to the area, of an infinitesimal triangle. This version of curvature is apparently due earlier to Gauss. The later more global version, of parallel transport, may be due later to Levi - Civita. In fact there is a much earlier Chinese mechanical device for producing it called the south pointing chariot.

http://en.wikipedia.org/wiki/South-pointing_chariot

Have you ever tried to prove that the angle sum of a triangle in the euclidean plane is π by translating a vector around the triangle, and noting that if you rotate it through the interior angle at each vertex so as to remain parallel to the new side, then when it returns to its original position, it has rotated exactly through π?

Notice that this process also gives π when performed on a sphere. But here it equals the (negative, i.e. clockwise) sum of the three interior angles through which one has translated, plus the contribution from parallel transporting the vectors around the three sides of the triangle, keeping them parallel to the geodesic sides all the way. Thus the discrepancy after parallel transport (called "holonomy") = angle sum $- \pi$. Hence the angle sum of the geodesic polynomial measures essentially the same phenomenon as does parallel transport around the geodesic polygon, i.e. average curvature of the interior of the polygon.

Thus the angle sum of polygons, which Gauss and Riemann knew reflected curvature, is equivalent to the total holonomy obtained by parallel transport. On a sphere where curvature is constant there is thus also a connection with area. You may see that if the radius, hence also curvature, of the sphere is 1, then the area of a geodesic triangle is equal to the angle sum $-\pi$. E.g. in post #19, if all three angles are right, this angle excess equals $\pi/2$, exactly the area of the geodesic triangle which in that case covers 1/8 of the sphere. If you expand the triangle to cover the entire sphere, the area equals $4\pi = 2\pi.2 = 2\pi.$(euler characteristic of the sphere).

In general you can triangulate a compact surface of genus $g > 0$ by $4g$ geodesic triangles, with $6g$ edges, $4g$ faces, and 2 vertices. Since there are only two vertices, the total angle sum of all the triangles is 4π. (E.g. a torus is triangulated by covering it by a rectangle with opposite edges identified, and then adding one vertex to the center of that rectangle and joining it to each vertex of the rectangle, obtaining 4 triangles, with 6 edges and 2 (distinct) vertices.)) If you then give the surface a metric of constant curvature -1, the area and angle sum formulas for the triangles adds up to give the formula area.(curvature) = $4g(\text{total angle sum} - \pi) = 4\pi - 4g\pi = 2\pi(2-2g) = 2\pi$.(euler characteristic of surface). This is called the Gauss - Bonnet formula.