In this last result we can take the limit $k \to 0$ and obtain an expansion for the Green function for (two-dimensional) polar coordinates:

$$\ln\left(\frac{1}{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}\right) = 2\ln\left(\frac{1}{\rho_>}\right) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos[m(\phi - \phi')] \tag{3.152}$$

This representation can be verified by a systematic construction of the two-dimensional Green function for the Poisson equation along the lines leading to (3.148). See Problem 2.17.

3.12 Eigenfunction Expansions for Green Functions

Another technique for obtaining expansions of Green functions is the use of eigenfunctions for some related problem. This approach is intimately connected with the methods of Sections 3.9 and 3.11.

To specify what we mean by eigenfunctions, we consider an elliptic differential equation of the form

same as the time-independent Schrodinger eqn

$$\nabla^2\psi(\mathbf{x}) + [f(\mathbf{x}) + \lambda]\psi(\mathbf{x}) = 0 \tag{3.153}$$

If the solutions $\psi(\mathbf{x})$ are required to satisfy homogeneous boundary conditions on the surface S of the volume of interest V, then (3.153) will not in general have well-behaved (e.g., finite and continuous) solutions, except for certain values of λ. These values of λ, denoted by λ_n, are called *eigenvalues* (or *characteristic values*) and the solutions $\psi_n(\mathbf{x})$ are called *eigenfunctions*.* The eigenvalue differential equation is written:

$$\nabla^2\psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda_n]\psi_n(\mathbf{x}) = 0 \tag{3.154}$$

By methods similar to those used to prove the orthogonality of the Legendre or Bessel functions, it can be shown that the eigenfunctions are orthogonal:

$$\int_V \psi_m^*(\mathbf{x})\psi_n(\mathbf{x})\, d^3x = \delta_{mn} \tag{3.155}$$

where the eigenfunctions are assumed normalized. The spectrum of eigenvalues λ_n may be a discrete set, or a continuum, or both. It will be assumed that the totality of eigenfunctions forms a complete set.

Suppose now that we wish to find the Green function for the equation:

$$\nabla_x^2 G(\mathbf{x}, \mathbf{x}') + [f(\mathbf{x}) + \lambda]G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{3.156}$$

where λ is *not* equal to one of the eigenvalues λ_n of (3.154). Furthermore, suppose that the Green function is to have the same boundary conditions as the eigenfunctions of (3.154). Then the Green function can be expanded in a series of the eigenfunctions of the form:

$$G(\mathbf{x}, \mathbf{x}') = \sum_n a_n(\mathbf{x}')\psi_n(\mathbf{x}) \tag{3.157}$$

*The reader familiar with wave mechanics will recognize (3.153) as equivalent to the Schrödinger equation for a particle in a potential.

The LHS here comes from inserting equation 3.157 into equation 3.156 and adding gamma_M and subtracting gamma_M for each eigenvalue, and then using equation 3.154 to eliminate many of these terms and leave behind this subtraction.

Substitution into the differential equation for the Green function leads to the result:

$$\sum_m a_m(\mathbf{x}')(\lambda - \lambda_m)\psi_m(\mathbf{x}) = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{3.158}$$

If we multiply both sides by $\psi_n^*(\mathbf{x})$ and integrate over the volume V, the orthogonality condition (3.155) reduces the left-hand side to one term, and we find:

$$a_n(\mathbf{x}') = 4\pi \frac{\psi_n^*(\mathbf{x}')}{\lambda_n - \lambda} \tag{3.159}$$

Consequently the eigenfunction expansion of the Green function is:

$$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n - \lambda} \tag{3.160}$$

For a continuous spectrum the sum is replaced by an integral.

Specializing the foregoing considerations to the Poisson equation, we place $f(\mathbf{x}) = 0$ and $\lambda = 0$ in (3.156). As a first, essentially trivial, illustration we let (3.154) be the wave equation over all space:

$$(\nabla^2 + k^2)\psi_\mathbf{k}(\mathbf{x}) = 0 \tag{3.161}$$

with the continuum of eigenvalues, k^2, and the eigenfunctions:

$$\psi_\mathbf{k}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.162}$$

In other words, these orthogonal functions are already normalized by the denominator of equation 3.162

These eigenfunctions have delta function normalization:

$$\int \psi_{\mathbf{k}'}^*(\mathbf{x})\psi_\mathbf{k}(\mathbf{x})\, d^3x = \delta(\mathbf{k} - \mathbf{k}') \tag{3.163}$$

Then, according to (3.160), the infinite space Green function has the expansion:

the discrete sum of eqn 3.160 has now become continuous

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{2\pi^2} \int d^3k\, \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2} \tag{3.164}$$

there's actually three differentials: dx, dy, dz

there's actually three integrands: x, y, z

This is just the three-dimensional Fourier integral representation of $1/|\mathbf{x} - \mathbf{x}'|$.

As a second example, consider the Green function for a Dirichlet problem inside a rectangular box defined by the six planes, $x = 0$, $y = 0$, $z = 0$, $x = a$, $y = b$, $z = c$. The expansion is to be made in terms of eigenfunctions of the wave equation:

This situation is similar to the problem of section 2.9 where the box had potential specified on it's faces.

$$(\nabla^2 + k_{lmn}^2)\psi_{lmn}(x, y, z) = 0 \tag{3.165}$$

where the eigenfunctions which vanish on all the boundary surfaces are

$$\psi_{lmn}(x, y, z) = \sqrt{\frac{8}{abc}} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi z}{c}\right)$$

and (3.166)

$$k_{lmn}^2 = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

The expansion of the Green function is therefore:

$$G(\mathbf{x}, \mathbf{x}') = \frac{32}{\pi abc} \times \sum_{l,m,n=1}^{\infty} \frac{\sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right) \sin\left(\frac{n\pi z}{c}\right) \sin\left(\frac{n\pi z'}{c}\right)}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \tag{3.167}$$

To relate expansion (3.167) to the type of expansions obtained in Sections 3.9 and 3.11, namely, (3.125) for spherical coordinates and (3.148) for cylindrical coordinates, we write down the analogous expansion for the rectangular box. If the x and y coordinates are treated in the manner of (θ, ϕ) or (ϕ, z) in those cases, while the z coordinate is singled out for special treatment, we obtain the Green function:

$$G(\mathbf{x}, \mathbf{x}') = \frac{16\pi}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right) \times \frac{\sinh(K_{lm}z_<) \sinh[K_{lm}(c - z_>)]}{K_{lm} \sinh(K_{lm}c)} \tag{3.168}$$

where $K_{lm} = \pi(l^2/a^2 + m^2/b^2)^{1/2}$. If (3.167) and (3.168) are to be equal, it must be that the sum over n in (3.167) is just the Fourier series representation on the interval $(0, c)$ of the one-dimensional Green function in z in (3.168):

$$\frac{\sinh(K_{lm}z_<) \sinh[K_{lm}(c - z_>)]}{K_{lm} \sinh(K_{lm}c)} = \frac{2}{c} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi z'}{c}\right)}{K_{lm}^2 + \left(\frac{n\pi}{c}\right)^2} \sin\left(\frac{n\pi z}{c}\right) \tag{3.169}$$

The verification that (3.169) is the correct Fourier representation is left as an exercise for the reader.

Further illustrations of this technique will be found in the problems at the end of the chapter.

3.13 Mixed Boundary Conditions; Conducting Plane with a Circular Hole

The potential problems discussed so far in this chapter have been of the orthodox kind in which the boundary conditions are of one type (usually Dirichlet) over the whole boundary surface. In the uniqueness proof for solutions of the Laplace or Poisson equation (Section 1.9) it was pointed out, however, that mixed boundary conditions, where the potential is specified over part of the boundary and its normal derivative is specified over the remainder, also lead to well-defined, unique boundary-value problems. Textbooks tend to mention the possibility of mixed boundary conditions when making the uniqueness proof and to ignore such problems in subsequent discussion. The reason, as we shall see, is that mixed boundary conditions are much more difficult to handle than the normal type.