Length Contraction with the Rod at Rest in S

Ad H.G.S. van der Ven

June 4, 2010

1 Derivation

As an effort for trying to understand Lorentz transformations, I'm trying to use them to derive the "time dilatation" result. Consider two reference frames, S (non-primed) and S' (primed), where S' is moving with respect to S with a velocity v.
Lorentz transformations:

$$x' = x\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} - vt\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} \tag{1}$$

and

$$t' = \left(t - \frac{vx}{c^2}\right)\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} \tag{2}$$

Now, I will try to derive the length contraction result. Suppose I have a thin rod situated in the *rest* frame, i.e. in the non-primed reference frame S. The left end of the rod is at x_1 and the right end of the rod is at x_2. Then, the length of the object in the rest frame or the non-primed reference frame is $L = x_2 - x_1$. I assume that $x_1 < x_2$. To find the corresponding coordinates of the ends of the rod in the primed frame x'_1 and x'_2, I will use (1). In order to do so I need to calculate $x'_2 - x'_1$ for the condition: $t'_1 = t'_2$. I am going to measure in the moving frame or primed frame and I need to mark the endpoints *simultaneously* in the primed frame. This means t'_2 equals t'_1 thus guaranteeing that t_2 is NOT equal to t_1.
Rewriting (2) with x_1 and t_1 instead of x and t results in:

$$t'_1 = \gamma(t_1 - vx_1/c^2) \tag{3}$$

Rewriting (2) with x_2 and t_2 instead of x and t results in:

$$t'_2 = \gamma(t_2 - vx_2/c^2) \tag{4}$$

In order to derive the length contraction I have to find the relation between t_1 and t_2, knowing that $t'_1 = t'_2$:

$$t'_1 = t'_2 \tag{5}$$

Substitution of (3) and (4) in (5) results in:

$$\gamma(t_1 - vx_1/c^2) = \gamma(t_2 - vx_2/c^2)$$
$$t_1 - vx_1/c^2 = t_2 - vx_2/c^2$$
$$t_2 - t_1 = \frac{v(x_2 - x_1)}{c^2} \tag{6}$$

Rewriting (1) with x_1 and t_1 instead of x and t results in:

$$x_1' \;=\; x_1 \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1} - vt_1 \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1} \tag{7}$$

Rewriting (1) with x_2 and t_2 instead of x and t results in:

$$x_2' \;=\; x_2 \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1} - vt_2 \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1} \tag{8}$$

We want to calculate

$$x_2' - x_1' \tag{9}$$

Substitution of (7) and (8) in (9) results in:

$$x_2' - x_1' = \gamma(x_2 - vt_2 - x_1 + vt_1)$$
$$x_2' - x_1' = \gamma[x_2 - x_1 - v(t_2 - t_1)] \tag{10}$$

Substitution of (6) in (10) results in:

$$x_2' - x_1' = \gamma \left[x_2 - x_1 - \frac{v^2(x_2 - x_1)}{c^2}\right]$$

$$x_2' - x_1' = \frac{x_2 - x_1}{\gamma}$$

which is the length contraction result. Now if we define l as

$$L = x_2 - x_1$$

and L' as

$$L' = x_2' - x_1'$$

then we obtain

$$L' = \frac{L}{\gamma}$$

where L represents the length of the rod as it would be measured in the rest frame, where it is located, and where L' represents the length of the rod as it would be measured from the moving frame. The same formula can be found in Wikipedia with the following defintion:

L is the proper length (the length of the object in its rest frame),
L' is the length observed by an observer in relative motion w.r.t. the object,
v is the relative velocity between the observer and the moving object,
c is the speed of light and
g the Lorentz factor.

However, **v** should be **-v**! Since S' is moving with velocity **v** relative to S.

2 Reference

To be written ...