Let $f(x) = \{ \begin{array}{c} 1,\ if\ x\ is\ rational. \\ 0, if\ x\ is\ irrational. \end{array} \}$. Prove $lim_{x \to 0} f(x)$ does not exist.

You start with the definition of limit: $lim_{x \to a} f(x) = \ell$ means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - a| < \delta$ then $|f(x) - \ell| < \epsilon$.

It's helpful to look at these definitions in as many differnet ways as possible so here is another form of the definition of limit: $lim_{x \to a} f(x) = \ell$ means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in (a - \delta, a + \delta)$ then $f(x) \in (\ell - \epsilon, \ell + \epsilon)$. All that we did here is to rephrase $|x - a| < \delta$ as $x \in (a - \delta, a + \delta)$. This says that a function $f(x)$ has a certain limit ℓ as x approaches some value a if given any (small) region (ϵ) around ℓ you can find some (small) region (δ) about x so that all the numbers in that small region around x are mapped by the function into the small region about ℓ.

Looking at your problem now, you want to find $lim_{x \to 0} f(x)$. Say that $lim_{x \to 0} f(x) = \ell$ where ℓ is some real number. Then you want to find a δ such that whenever $x \in (a - \delta, a + \delta) = (-\delta, \delta)$ we have $f(x) \in (\ell - \epsilon, \ell + \epsilon)$. And this has to be true for every ϵ. (In this sense you don't pick your ϵ). Now, for any δ you note that there will be both irrational numbers and rational numbers in the interval $(-\delta, \delta)$ so $f(x)$ will equal both 1 and 0. So what δ is there that all the numbers in that interval $(-\delta, \delta)$ will all be mapped by this function to a very small interval around some real number ℓ?