(i) DEFINITION:

$[a, b] \stackrel{\text{def}}{=}$ closed interval in x-axis

$[c, d] \stackrel{\text{def}}{=}$ closed interval in y-axis

$Q \stackrel{\text{def}}{=} [a, b] \times [c, d]$

$P_x \stackrel{\text{def}}{=}$ any partition of $[a, b] = \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$ $[n \in \mathbb{N}]$

$P_y \stackrel{\text{def}}{=}$ any partition of $[c, d] = \{c = y_0, y_1, \dots, y_{m-1}, y_m = d\}$ $[m \in \mathbb{N}]$

$P \stackrel{\text{def}}{=} P_x \times P_y \stackrel{\text{def}}{=} \{(x_0, y_0), (x_0, y_1), (x_0, y_2), \dots, (x_0, y_m),$

$(x_1, y_0), (x_1, y_1), (x_1, y_2), \dots, (x_1, y_m),$

....

$(x_n, y_0), (x_n, y_1), (x_n, y_2), \dots, (x_n, y_m)\}$

$Q_{ij} \stackrel{\text{def}}{=} (x_{i-1}, x_i) \times \left(y_{j-1}, y_j\right)$ $[(n \geq i \geq 1) \in \mathbb{N}, (m \geq j \geq 1) \in \mathbb{N}]$

¶ $\stackrel{\text{def}}{=}$ set of all possible P

P^* is refinement of $P \stackrel{\text{def}}{\Leftrightarrow} P \subseteq P^*$

Б: $Q \longrightarrow \mathbb{R} = \mathrm{SF}$ $\stackrel{\text{def}}{\Leftrightarrow} \exists\ P \ni \forall\ Q_{ij}\ , \left(\text{Б}: Q_{ij} \longrightarrow \mathbb{R}\right) = ©_{ij}$ ^ Б: partition boundary $\longrightarrow \mathbb{R}$

(ii) DEFINITION:

$f_{ij} = f : Q_{ij} \longrightarrow \mathbb{R}$

Lemma 1:

$$[f = \mathrm{SF}_1\,] \Longrightarrow [c_1\, f = \mathrm{SF}_2]$$

Proof:

$f = \mathrm{SF}_1$

{given}

$\Longrightarrow \exists\ P$ (choose P_a) $\ni \forall\ Q_{ij}\ , f = ©_{ij}\ \wedge\ f :$ partition boundary $\longrightarrow \mathbb{R}$

{definition of SF}

$\Longrightarrow$ For P_a, $\forall\ Q_{ij}\ , c_1\ f = c_1\ ©_{ij}\ \wedge\ c_1\ f :$ partition boundary $\longrightarrow \mathbb{R}$

{Fig 1}

$\Longrightarrow \exists\ P \ni \forall\ Q_{ij}\ , c_1\ f = c_1\ ©_{ij}\ \wedge\ c_1\ f :$ partition boundary $\longrightarrow \mathbb{R}$

{there exist}

$\Longrightarrow c_1\ f = \mathrm{SF}_2$

{definition of SF}

(iii) DEFINITION:

$$P_{f\,x} \stackrel{\text{def}}{=} \text{any partition of } [a,b] = \{a = x_0{}^f, x_1{}^f, \dots, x_{\alpha-1}{}^f, x_\alpha{}^f = b\} \qquad [\alpha \in \mathbb{N}]$$

$$P_{f\,y} \stackrel{\text{def}}{=} \text{any partition of } [c,d] = \{c = y_0{}^f, y_1{}^f, \dots, y_{\beta-1}{}^f, y_\beta{}^f = d\} \qquad [\beta \in \mathbb{N}]$$

$$P_{g\,x} \stackrel{\text{def}}{=} \text{any partition of } [a,b] = \{a = x_0{}^g, x_1{}^g, \dots, x_{\xi-1}{}^g, x_\xi{}^g = b\} \qquad [\xi \in \mathbb{N}]$$

$$P_{g\,y} \stackrel{\text{def}}{=} \text{any partition of } [c,d] = \{c = y_0{}^g, y_1{}^g, \dots, y_{\eta-1}{}^g, y_\eta{}^g = d\} \qquad [\eta \in \mathbb{N}]$$

$$P_x \stackrel{\text{def}}{=} P_{f\,x} \cup P_{g\,x} = \{a = x_0{}^*, x_1{}^*, \dots, x_{\text{ख}-1}{}^*, x_{\text{ख}}{}^* = b\} \qquad [\text{ख} \in \mathbb{N}]$$

$$P_y \stackrel{\text{def}}{=} P_{f\,y} \cup P_{g\,y} = \{c = y_0{}^*, y_1{}^*, \dots, y_{\text{ग}-1}{}^*, y_{\text{ग}}{}^* = d\} \qquad [\text{ग} \in \mathbb{N}]$$

$$\begin{aligned} P_x \times P_y = \{&(x_0{}^*, y_0{}^*), (x_0{}^*, y_1{}^*), (x_0{}^*, y_2{}^*), \dots, (x_0{}^*, y_{\text{ग}}{}^*),\\ &(x_1{}^*, y_0{}^*), (x_1{}^*, y_1{}^*), (x_1{}^*, y_2{}^*), \dots, (x_1{}^*, y_{\text{ग}}{}^*),\\ &\dots\dots\\ &(x_{\text{ख}}{}^*, y_0{}^*), (x_{\text{ख}}{}^*, y_1{}^*), (x_{\text{ख}}{}^*, y_2{}^*), \dots, (x_{\text{ख}}{}^*, y_{\text{ग}}{}^*)\} \end{aligned}$$

$$Q_{ij} \stackrel{\text{def}}{=} (x_{i-1}, x_i) \times (y_{j-1}, y_j) \qquad [(\text{ख} \geq i \geq 1) \in \mathbb{N}, (\text{ग} \geq j \geq 1) \in \mathbb{N}]$$

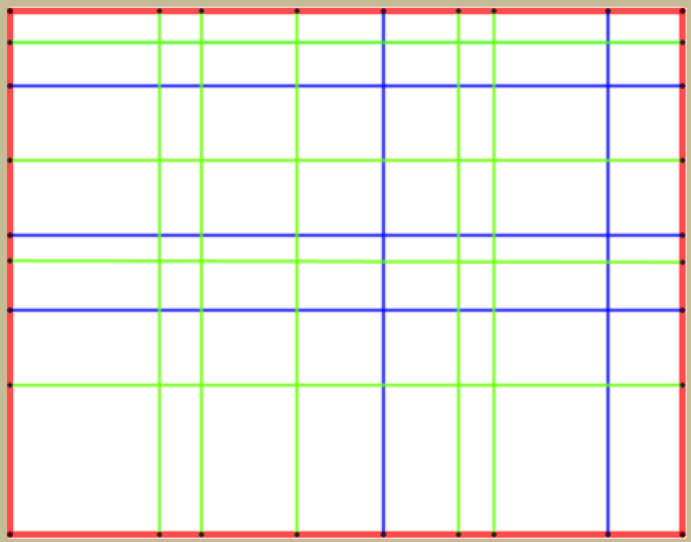


FIG 1

Theorem 1:

$[f = \mathrm{SF}_1 \wedge g = \mathrm{SF}_2] \Longrightarrow [c_1 f + c_2 g = \mathrm{SF}_3]$

Proof:

$f = \mathrm{SF}_1 \wedge g = \mathrm{SF}_2$

{given}

$\Longrightarrow c_1 f = \mathrm{SF}_3 \wedge c_2 g = \mathrm{SF}_4$

{**Theorem 1**}

$\Longrightarrow$ For $(P_x \times P_y) \in ¶$, $\forall Q_{ij}$, $(c_1 f + c_2 g : Q_{ij} \longrightarrow \mathbb{R}) = ©_{ij} \wedge c_1 f + c_2 g :$ partition boundary $\longrightarrow \mathbb{R}$

{Fig 1}

$\Longrightarrow \exists (P_x \times P_y) \in ¶ \ni \forall Q_{ij}$, $(c_1 f + c_2 g : Q_{ij} \longrightarrow \mathbb{R}) = ©_{ij} \wedge c_1 f + c_2 g :$ partition boundary $\longrightarrow \mathbb{R}$

{there exist}

$\Longrightarrow c_1 f + c_2 g = \mathrm{SF}_3$

{definition of SF}

(iv) DEFINITION:

$P_x{}^* \overset{\text{def}}{\equiv}$ a refinement of $[a, b] = \{a = x_0, x_1, \dots x_{a-1}, x^*, x_a, \dots x_{n-1}, x_n = b\}$ $[n \in \mathbb{N}]$

$P_y{}^* \overset{\text{def}}{\equiv}$ a refinement of $[c, d] = \{c = y_0, y_1, \dots, y_{m-1}, y_m = d\}$ $[m \in \mathbb{N}]$

$$P^* \overset{\text{def}}{\equiv} P_x{}^* \times P_y{}^* \overset{\text{def}}{\equiv} \{(x_0, y_0), (x_0, y_1), (x_0, y_2), \dots, (x_0, y_m),$$
$$(x_1, y_0), (x_1, y_1), (x_1, y_2), \dots, (x_1, y_m),$$
$$\dots\dots\dots$$
$$(x_n, y_0), (x_n, y_1), (x_n, y_2), \dots, (x_n, y_m)\}$$

Lemma 2:

$$Б: Q \longrightarrow \mathbb{R} = \mathrm{SF}_Б \overset{\text{def}}{\Leftrightarrow} \exists\, P \in ¶ \ni \forall\, Q_{ij}\,, \big(Б: Q_{ij} \longrightarrow \mathbb{R}\big) = ©_{ij} \wedge Б\text{: partition boundary} \longrightarrow \mathbb{R}$$
$$ɖ: Q \longrightarrow \mathbb{R} = \mathrm{SF}_ɖ \overset{\text{def}}{\Leftrightarrow} \exists\, (P^* \supseteq P) \in ¶ \ni \forall\, Q_{kl}\,, \big(ɖ: Q_{ij} \longrightarrow \mathbb{R}\big) = ©_{ij} \wedge ɖ\text{: partition boundary} \longrightarrow \mathbb{R}$$

$$\Longrightarrow$$

$$\iint_Q Б = \iint_Q ɖ$$

Proof:

$Б: Q \longrightarrow \mathbb{R} = \mathrm{SF}_Б$

$$\Longrightarrow \iint_Q Б \overset{\text{def}}{\equiv} \sum_{j=1}^{m} \sum_{i=1}^{n} ©_{ij}\, (x_i - x_{i-1})\, (y_j - y_{j-1})$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{a-1} ©_{ij}\, (x_i - x_{i-1})\, (y_j - y_{j-1})$$

$$+ \sum_{j=1}^{m} ©_{aj}\, (x_a - x_{a-1})\, (y_j - y_{j-1})$$

$$+ \sum_{j=1}^{m} \sum_{i=a+1}^{n} ©_{ij}\, (x_i - x_{i-1})\, (y_j - y_{j-1})$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{a-1} ©_{ij} \, (x_i - x_{i-1}) \, (y_j - y_{j-1})$$

$$+ \sum_{j=1}^{m} ©_{aj} \, (x^* - x_{a-1}) \, (y_j - y_{j-1})$$

$$+ \sum_{j=1}^{m} ©_{aj} \, (x_a - x^*) \, (y_j - y_{j-1})$$

$$+ \sum_{j=1}^{m} \sum_{i=a+1}^{n} ©_{ij} \, (x_i - x_{i-1}) \, (y_j - y_{j-1}) \tag{1}$$

$$ɖ\colon Q \longrightarrow \mathbb{R} = \mathrm{SF}_{ɖ}$$

$$\Longrightarrow \iint_Q ɖ \quad \overset{\text{def}}{\equiv} \sum_{j=1}^{m} \sum_{i=1}^{a-1} ©_{ij} \, (x_i - x_{i-1}) \, (y_j - y_{j-1})$$

$$+ \sum_{j=1}^{m} ©_{aj} \, (x^* - x_{a-1}) \, (y_j - y_{j-1})$$

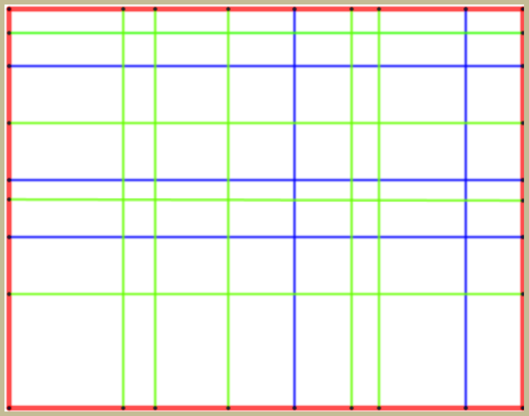
$$+ \sum_{j=1}^{m} ©_{aj} \, (x_a - x^*) \, (y_j - y_{j-1})$$

$$+ \sum_{j=1}^{m} \sum_{i=a+1}^{n} ©_{ij} \, (x_i - x_{i-1}) \, (y_j - y_{j-1})$$

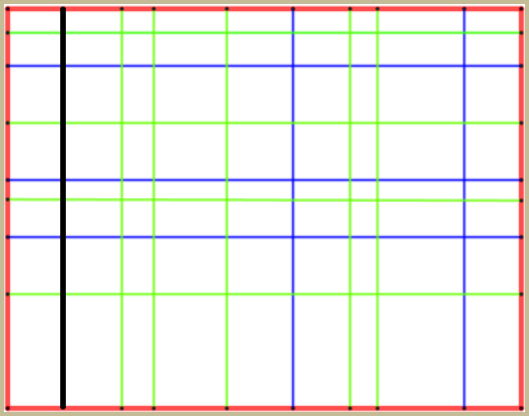
$$= \iint_Q \text{Б}$$

{by (1)}

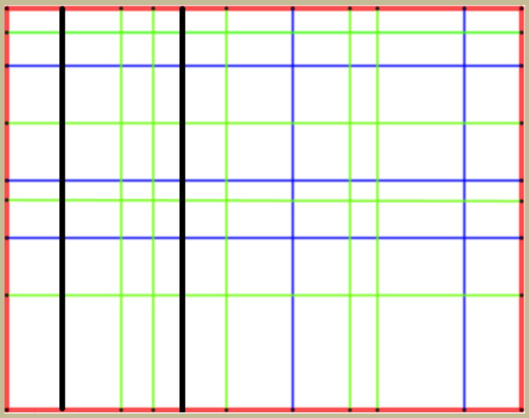
(v) **DEFINITION:**



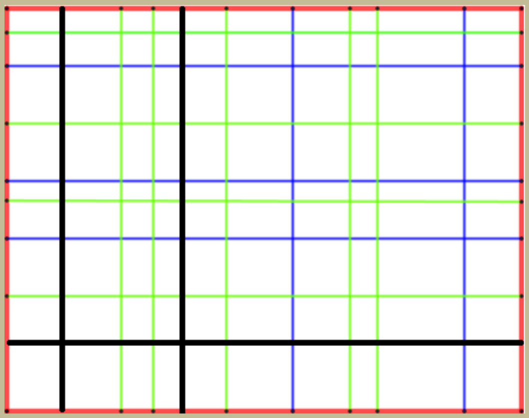
A



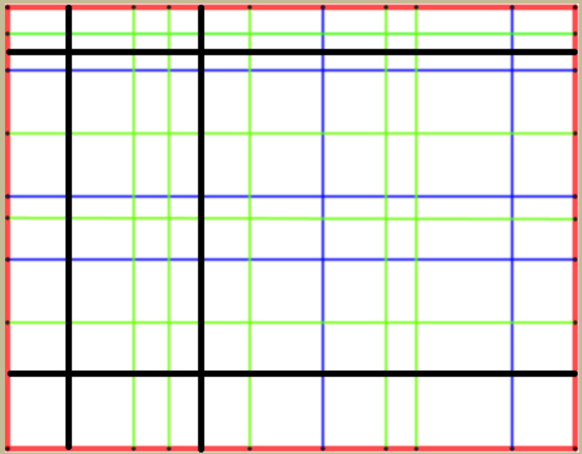
B



C



D



E

$P_A \stackrel{\text{def}}{=}$ set of all "intersecting points in A"

$P_B \stackrel{\text{def}}{=}$ set of all "intersecting points in B"

$P_C \stackrel{\text{def}}{=}$ set of all "intersecting points in C"

$P_D \stackrel{\text{def}}{=}$ set of all "intersecting points in D"

$P_E \stackrel{\text{def}}{=}$ set of all "intersecting points in E"

$\iint_{Q/P_A}$ Б $\stackrel{\text{def}}{=} \iint_Q$ Б using partition P_A

$\iint_{Q/P_B}$ Б $\stackrel{\text{def}}{=} \iint_Q$ Б using partition P_B

$\iint_{Q/P_C}$ Б $\stackrel{\text{def}}{=} \iint_Q$ Б using partition P_C

$\iint_{Q/P_D}$ Б $\stackrel{\text{def}}{=} \iint_Q$ Б using partition P_D

$\iint_{Q/P_E}$ Б $\stackrel{\text{def}}{=} \iint_Q$ Б using partition P_E

Theorem 2:

Б: $Q \longrightarrow \mathbb{R} = \mathrm{SF} \implies \forall\ (P \in ¶)$, $\iint_Q$ Б remains unaltered

$$\iint_{Q/P_A} Б = \iint_{Q/P_B} Б = \iint_{Q/P_C} Б = \iint_{Q/P_D} Б = \iint_{Q/P_E} Б$$

{Lemma 2}

By exact same logic, $\forall\ (P \in ¶)$, $\iint_Q$ Б remains unaltered.

(vi) DEFINITION:

(vii) DEFINITION: