The Full Mathematics of:

SPECIAL Relativity

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CONTENTS

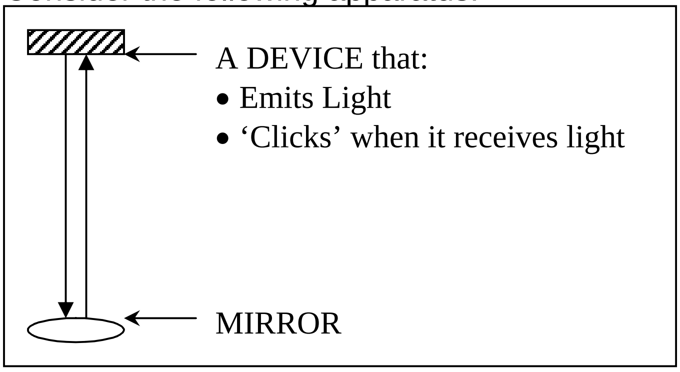
INTRODUCTION

This document uses a straightforward route to understand the derivation of certain concepts of Special Relativity. Note that this document does not use the idea of 4 Vectors in order to derive any of the equations…

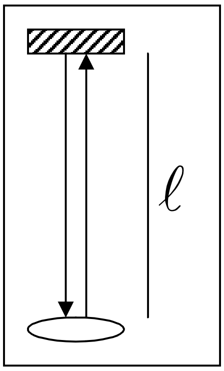
Before Einstein's theories, it was believed that light was a wave that required a medium. This medium was believed to be Ether. Light did not have a limit. However, Michealson and Morley conducted an experiment that was repeated again and again in the aim to attain a result that showed that viewed the speed of light to move faster. However, the experiment yielded a constant value, no matter how faster the observer was moving. So Einstein came up with Special Relativity to explain this; he had to modify conventional views of time and space to accommodate for this constancy of the speed of light.

TIME DILATION & LENGTH CONTRACTION

Consider the following apparatus:



The time for 1 'click' is given by:

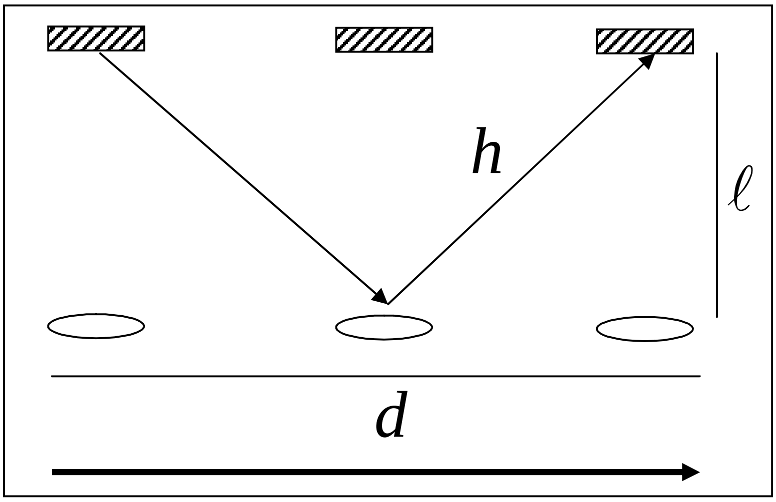


$$t' = \frac{d}{v}$$

$$\therefore t' = \frac{2\ell}{c}$$

$$\therefore \ell = \frac{ct'}{2} \rightarrow$$

Now; if the apparatus moves to the right, the time for 1 'Click' from the viewpoint of a Stationary observer is given by (notice the diagonal path taken by the light beam):



$$t = \frac{d}{v}$$

$$\therefore t = \frac{2h}{c}$$

$$\therefore h = \frac{ct}{2} \rightarrow$$

Also:

$$v = \frac{\delta d}{\delta t}$$

Integrating with respect to t

$$\therefore vt = d \rightarrow$$

Solving Mathematically:

By: Pythagoras Theorem

$$\therefore h^2 = \left(\frac{d}{2}\right)^2 + \ell^2$$

Substituting

$$\therefore \left(\frac{ct}{2}\right)^2 = \left(\frac{vt}{2}\right)^2 + \left(\frac{ct'}{2}\right)^2$$

$$\therefore c^2t^2 = v^2t^2 + c^2t'^2$$

$$\therefore c^2t'^2 = c^2t^2 - v^2t^2$$

$$\therefore t'^2 = t^2 - \frac{v^2t^2}{c^2}$$

$$\therefore t'^2 = t^2\left(1 - \frac{v^2}{c^2}\right)$$

$$\therefore t = \frac{t'}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow$$

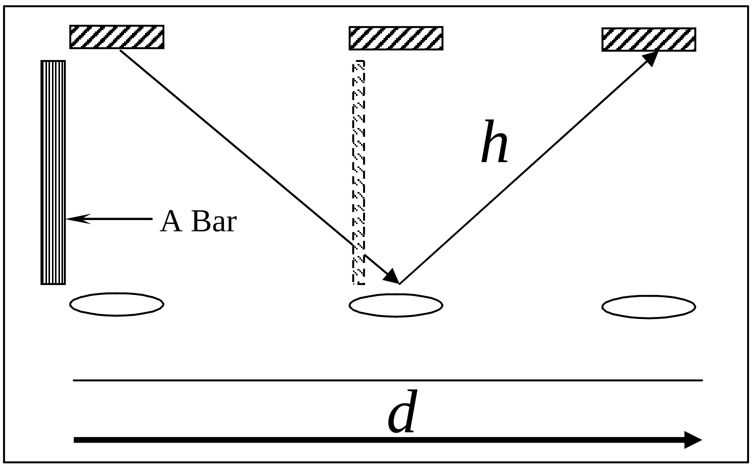
This yields the TIME DILATION EQUATION.

Now consider the same apparatus except with a bar adjoining the mirror and the device:

let:

$\ell_0 =$ Original Length (of the bar) in a Stationary frame of reference

$\ell =$ Length (of the bar) in a Moving Frame of Reference



$$t' = \frac{2\ell_0}{c}$$

$$\therefore \frac{t}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\ell_0}{c}$$

$$\therefore t = \frac{2\ell_0\sqrt{1-\frac{v^2}{c^2}}}{c} \rightarrow$$

$$t = \frac{2h}{c} \equiv \frac{2\ell}{c} \rightarrow$$

equating the 2 results

$$\therefore \frac{2\ell}{c} = \frac{2\ell_0\sqrt{1-\frac{v^2}{c^2}}}{c}$$

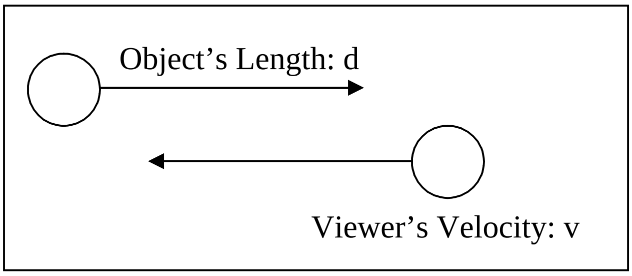
$$\therefore \ell = \ell_0\sqrt{1-\frac{v^2}{c^2}} \Rightarrow$$

This yields the equation for LENGTH CONTRACTION.

LORENZ TRANSFORMATIONS & COMPOSITION OF VELOCITIES

The Lorenz Transformations can be thought of as conversion of certain Physical Quantities involving time, between Frames.

For example, in classical physics, if an object's velocity is being measured, the velocity of the reference frame is added to the velocity seen:



The viewer measures the length, d as:

$d' = d + vt$

BUT if $v = c$

$d' \neq d + vt$

$\because d' > c$

The Lorenz Transformations are obviously more complex, but they first take the same form:

$$x' = \alpha x + \beta t$$
$$t' = \varepsilon x + \eta t$$
$$where:$$
$$x = distance$$
$$\alpha, \beta, \varepsilon, \eta = constants\ involving\ the\ reference\ frame...$$

In the beginning: $t = 0,\ x = 0$. This is because the 2 Frames start from the origin. The origin (0,0) of each Frame of Reference coincides together in the same place in 2-Dimensional Space.

$\therefore t = t' = 0$ The situation, just when the 2 Clocks are started...

$\therefore x = x' = 0$

$\because \varepsilon x = 0$

Solving Mathematically:

$$\because x' = 0$$
$$\therefore 0 = \alpha(vt) + \beta t$$
$$\therefore b = -\alpha v$$

Substituting into $x' = \alpha x + \beta t$

$$\therefore x' = \alpha(x - vt) \rightarrow$$

Note that this is very similar to the Classical theory, except for the multiplier α. Also, since 'the laws of Physics look the same in all reference frames':

$$\therefore x = \alpha(x' + vt') \rightarrow$$

Substituting:

$$\therefore x = \alpha(\alpha(x - vt) + vt')$$
$$\therefore x = \alpha^2 x - \alpha^2 vt + \alpha vt'$$
$$\therefore x(1 - \alpha^2) + \alpha^2 vt = \alpha vt'$$
$$\therefore t' = \alpha t - \frac{x(\alpha^2 - 1)}{av} \rightarrow$$

The time taken of a certain event in recorded in the frames is given by:

$$t_2 - t_1 = \alpha\left(t'_2 - t'_1\right)$$

If it were an instantaneous event:

$$t = \alpha t'$$

By the Time Dilation Equation:

$$t = \gamma t'$$

$$\therefore \alpha = \gamma \rightarrow$$

Substituting:

$$\therefore x' = \gamma\left(x - vt\right) \Rightarrow$$

$$\therefore y' = y \Rightarrow$$

$$\therefore z' = z \Rightarrow$$

$\dfrac{\gamma^2 - 1}{\gamma}$ or $\dfrac{\alpha^2 - 1}{\alpha}$

$$= \frac{\left(\frac{1}{1 - \frac{v^2}{c^2}} - 1\right)}{\gamma}$$

$$= \frac{\left(\frac{1 - \left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2}}\right)}{\gamma}$$

$$= \frac{\left(\frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}\right)}{\gamma}$$

$$= \frac{v^2}{c^2\left(1 - \frac{v^2}{c^2}\right)\gamma}$$

$$= \frac{v^2\gamma^2}{c^2\gamma}$$

$$= \frac{v^2\gamma}{c^2} \rightarrow$$

Since: $t' = \alpha t - \dfrac{x\left(\alpha^2 - 1\right)}{av}$

Substituting:

$$\therefore t' = \alpha t - \frac{\gamma x v}{c^2}$$

$$\therefore t' = \gamma\left(t - \frac{xv}{c^2}\right) \Rightarrow$$

Since $v = \dfrac{\delta d}{\delta t}$, the Distance Transformations and the Time Transformations should be modified like in any situation, where values are taken for a rate of change. Hence, the smallest infinitesimal values must be used to express these equations:

$$\therefore \delta x' = \gamma\left(\delta x - v\delta t\right) \rightarrow$$
$$\therefore \delta y' = \delta y \rightarrow$$
$$\therefore \delta z' = \delta z \rightarrow$$
$$\therefore \delta t' = \gamma\left(\delta t - \frac{v\delta x}{c^2}\right) \rightarrow$$

Now all the equations are factorised by a common factor of δt.

$$u_x' = \frac{\delta x'}{\delta t'} \rightarrow$$
$$\therefore u_x' = \frac{\gamma\left(\delta x - v\delta t\right)}{\gamma\left(\delta t - \frac{v\delta x}{c^2}\right)}$$
$$\therefore u_x' = \frac{\delta t\left(\frac{\delta x}{\delta t} - v\right)}{\delta t\left(1 - \frac{v\delta x}{c^2\delta t}\right)}$$

Since: $u_x = \frac{\delta x}{\delta t}$

$$\therefore u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \Rightarrow$$
$$u_y' = \frac{\delta y'}{\delta t'} \rightarrow$$
$$\therefore u_y' = \frac{\delta y}{\gamma\left(\delta t - \frac{v\delta x}{c^2}\right)}$$
$$\therefore u_y' = \frac{\delta t\left(\frac{\delta y}{\delta t}\right)}{\gamma\delta t\left(1 - \frac{v\delta x}{c^2\delta t}\right)}$$

Since: $u_y = \frac{\delta y}{\delta t}$

$$\therefore u_y' = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \Rightarrow$$
$$u_z' = \frac{\delta z'}{\delta t'} \rightarrow$$
$$\therefore u_z' = \frac{\delta z}{\gamma\left(\delta t - \frac{v\delta z}{c^2}\right)}$$
$$\therefore u_z' = \frac{\delta t\left(\frac{\delta z}{\delta t}\right)}{\gamma\delta t\left(1 - \frac{v\delta z}{c^2\delta t}\right)}$$

Since: $u_z = \frac{\delta z}{\delta t}$

$$\therefore u_z' = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \Rightarrow$$

One of the famous equations derived from the first of the above equations is known as the VELOCITY COMPOSITION LAW:

$$\therefore u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$
$$\therefore u_x' - \frac{u_x v\left(u_x'\right)}{c^2} = u_x - v$$

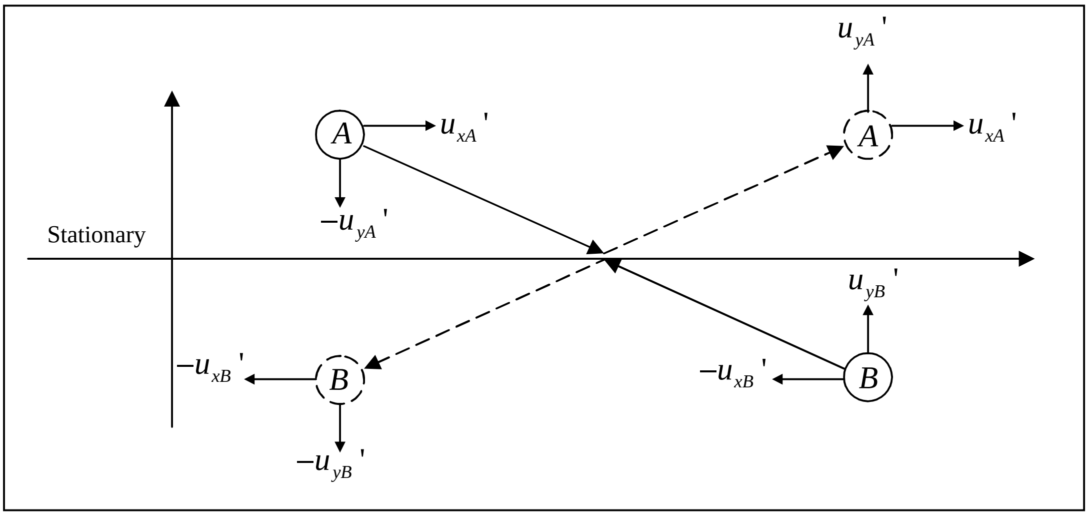
$$\therefore u_x{}' + v = u_x\left(1 + \frac{v u_x{}'}{c^2}\right)$$

$$\therefore u_x = \frac{u_x{}' + v}{1 + \frac{v u_x{}'}{c^2}}$$

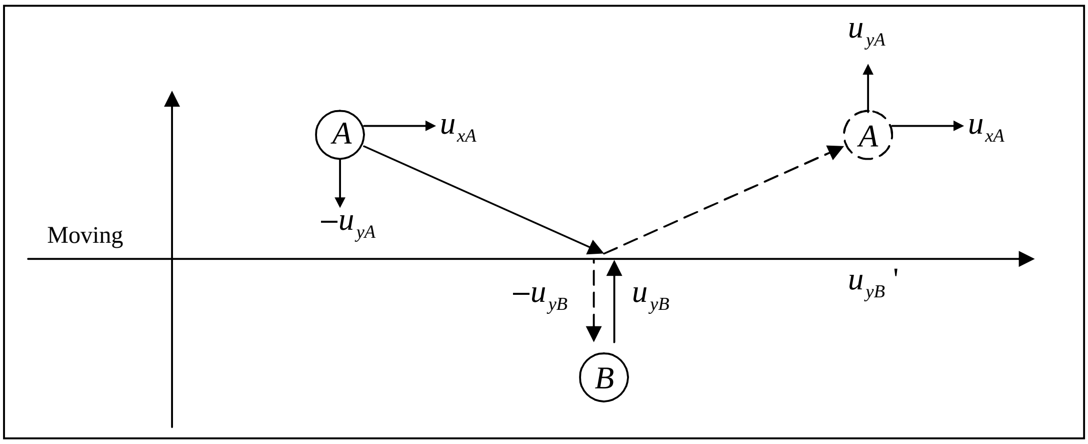
$$\therefore w = \frac{u + v}{1 + \frac{uv}{c^2}} \Rightarrow$$

RELATIVISTIC MASS

Consider the following diagram below: A stationary observer observes a collision between, Particle A and Particle B:



Now the same collision is viewed from a moving reference frame. This frame moves at the same velocity as Particle B:



Using the Velocity transformation Equations:

$$\therefore u_{yA}' = \frac{u_{yA}}{\gamma\left(1-\frac{vu_{xA}}{c^2}\right)} \rightarrow$$

Since $u_{xB} = 0$

$$\therefore u_{yB}' = \frac{u_{yB}}{\gamma} \rightarrow$$

If $u_{yA}' = u_{yB}'$ and the masses of the particles were the same it would yield:

$$\Delta P_B = \Delta P_A$$

$$\therefore 2m\left(u_{yB}'\right) = 2m\left(u_{yA}'\right) \rightarrow$$

$$\therefore u_{yB} = \frac{u_{yA}}{1-\frac{vu_{xA}}{c^2}}$$

$u_{yB} \neq u_{yA}$ As opposed to Newtonian mechanics

Thus Momentum is not conserved in relative frames. So if the masses were named: M_A and M_B:

$$\therefore M_A = M_B \cdot \frac{u_{yB}}{u_{yA}} \rightarrow$$

Since: $u_{yB} = \dfrac{u_{yA}}{1-\frac{vu_{xA}}{c^2}}$

$$\therefore \frac{u_{yB}}{u_{yA}} = \frac{1}{1-\frac{vu_{xA}}{c^2}}$$

Substituting:

$$\therefore M_A = \frac{M_B}{1-\frac{vu_{xA}}{c^2}} \rightarrow$$

This shows that if the principle of relativity is to apply; the mass must change by an amount such that the conservation of momentum is true.

NOTE: The following sets of equations will use the Velocity Transformation Law that applies for horizontal velocities:

$$u_x' = \frac{u_x - v}{1-\frac{u_x v}{c^2}}$$

In the stationary reference frame *(refer to diagram)* it is clear that: $u'_{yA} = -u'_{yB}$:

$$\therefore u'_{xB} = \frac{u_{xB} - v}{1-\frac{u_{xB} v}{c^2}}$$

$$\therefore u'_{xB} = \frac{0 - v}{1-\frac{0v}{c^2}}$$

$$\therefore u'_{xB} = -v$$

$$\therefore u'_{xA} = v \rightarrow$$

Now v can be expressed in terms of u_{xA} …

$$\therefore u'_{xA} = \frac{u_{xA} - v}{1-\frac{u_{xA} v}{c^2}} = v$$

$$\therefore u_{xA} - v = v - \frac{u_{xA} v^2}{c^2}$$

$$\therefore u_{xA}c^2 - 2vc^2 + u_{xA}v^2 = 0$$

$$\therefore c^2 - \frac{2c^2}{u_{xA}} v + v^2 = 0$$

completing the square...

$$\therefore \left(v - \frac{c^2}{u_{xA}}\right)^2 - \left(\frac{c^2}{u_{xA}}\right)^2 + c^2 = 0$$

$$\therefore \left(v - \frac{c^2}{u_{xA}}\right) = \pm\sqrt{\frac{c^4}{u_{xA}^2} - c^2}$$

$$\therefore \left(v - \frac{c^2}{u_{xA}}\right) = \pm\sqrt{\frac{c^4}{u_{xA}^2}\left(1 - \frac{u_{xA}^2}{c^2}\right)}$$

$$\therefore v = \frac{c^2}{u_{xA}} \pm \frac{c^2}{u_{xA}}\sqrt{\left(1 - \frac{u_{xA}^2}{c^2}\right)}$$

$$\therefore v = \frac{c^2}{u_{xA}}\left(1 - \sqrt{1 - \frac{u_{xA}^2}{c^2}}\right) \rightarrow$$ The plus is ignored as it is unneccessary later

Now this equation is substituted into $M_A = \frac{M_B}{1 - \frac{vu_{xA}}{c^2}}$:

$$\therefore M_A = \frac{M_B}{1 - \frac{u_{xA}}{c^2} \cdot \frac{c^2}{u_{xA}}\left(1 - \sqrt{1 - \frac{u_{xA}^2}{c^2}}\right)}$$

$$\therefore M_A = \frac{M_B}{1 - 1 + \sqrt{1 - \frac{u_{xA}^2}{c^2}}}$$

$$\therefore M_A = \frac{M_B}{\sqrt{1 - \frac{u_{xA}^2}{c^2}}} \rightarrow$$

Now Object B is at rest (in the moving frame), thus its mass is known as the 'Rest Mass' with the symbol m_0.

Now in the beginning the masses were identical (i.e. when there was no velocity). But as they started to move, Object B would perceive Object A's mass differently, this is due to the relative velocity it is travelling in – this is known as 'Relativistic Mass' symbolized as: m.

Since the velocity given in the above equation corresponds to an observer who is stationary with the Object B (i.e. 'moves with it'), it is fair to label it: u.

$$\therefore m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

The more convenient notation is:

$$\therefore m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow$$

RELATIVISTIC ENERGY

There are many ways to reach the final Equation – some involving the Binomial Theorem of Expansion. But, here is a proof that just uses simple differential calculus. It starts off with the equation for Relativistic Mass:

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \rightarrow$$

$$\therefore m = \frac{\delta m}{\delta v} \cdot \frac{\delta v}{\delta m} \cdot m \rightarrow \qquad \because \frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta y} = 1$$

$$\therefore \frac{\delta m}{\delta v} = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \cdot \frac{-1}{2} \cdot \frac{-2v}{c^2} \quad \text{by chain rule}$$

$$\therefore \frac{\delta m}{\delta v} = \frac{m_0 v}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}} \cdot c^2} \rightarrow$$

$$\therefore \frac{\delta v}{\delta m} = \frac{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}} \cdot c^2}{m_0 v}$$

Substituting m_0 with $m\sqrt{1-\frac{v^2}{c^2}}$

$$\therefore \frac{\delta v}{\delta m} = \frac{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}} \cdot c^2}{mv\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \frac{\delta v}{\delta m} = \frac{\left(1-\frac{v^2}{c^2}\right) \cdot c^2}{mv}$$

$$\therefore \frac{\delta v}{\delta m} = \frac{c^2 - v^2}{mv} \rightarrow$$

Since: $m = \frac{\delta m}{\delta v} \cdot \frac{\delta v}{\delta m} \cdot m$

$$\therefore m = \frac{\delta m}{\delta v} \cdot \frac{c^2 - v^2}{mv} \cdot m$$

$$\therefore m = \frac{\delta m}{\delta v} \cdot \frac{c^2 - v^2}{v} \rightarrow$$

$$F = ma = \frac{mv}{t} = \frac{\rho}{t} \rightarrow$$

$$\rho = mv$$

$$\therefore \frac{\delta \rho}{\delta t} = \frac{\delta(mv)}{\delta t}$$

$$\therefore F = m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t}$$

Substituting m

$$\therefore F = v\frac{\delta m}{\delta t} + \frac{\delta v}{\delta t}\left(\frac{\delta m}{\delta v} \cdot \frac{c^2 - v^2}{v}\right)$$

$$\therefore F = v\frac{\delta m}{\delta t} + \frac{\delta m}{\delta t}\frac{c^2 - v^2}{v} \qquad \because \frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$$

Seperation of Variables

$$\therefore F\delta t = v\delta m + \delta m\left(\frac{c^2 - v^2}{v}\right)$$

$$\therefore F\delta t \cdot v = v^2\delta m + \delta m\left(c^2 - v^2\right) \rightarrow$$

Since: $\delta E = F\delta d \equiv F\delta t \cdot v$

$$\therefore \delta E = \left(v^2 + c^2 - v^2\right)\delta m$$

$$\therefore \delta E = c^2\delta m$$

$$\therefore \frac{\delta E}{\delta m} = c^2$$

$$\therefore \int\left(\frac{\delta E}{\delta m}\right)\delta m = \int\left(c^2\right)\delta m$$

$$\therefore E = mc^2$$

Substituting the equation of Relativistic Mass

$$\therefore E = \gamma m_0 c^2$$

$$\therefore E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow$$

if $v = 0$ - i.e. at rest:

$$E = m_0 c^2 \Rightarrow$$

BIBLIOGRAPHY

Wiki-Books – “Special Relativity Dynamics”

GLOSSARY

DISTANCE: Distance is a basic Fundamental SI Unit. It cannot be defined. However, it can be thought of as the measure of the amount of space between 2 points.

EINSTEIN'S FIRST POSTULATE: "The Laws of Physics are the same in all Reference Frames" or "There is no Universal Reference Frame"

EINSTEIN'S SECOND POSTULATE: "The velocity of the speed of light in free space is a constant in all Reference Frames"

ENERGY: Work is done when a force acts upon an Object. Energy is the capacity to do work. However, Einstein showed that an object need not do work in order to possess any energy – if the object has any mass, then it can have Energy.

FORCE: The action or influence that causes that acceleration or changes in shape of an object.

INERTIA: A property of Matter that causes it to resist any change in Velocity or Direction.

LENGTH CONTRACTION: The shrinkage in length observed when an observer in a stationary frame observes a moving object.

MASS: Mass is a Physical Quantity that measures the amount of Inertia a body contains.

MATTER: Anything that occupies free space and has attributes of Gravity and Inertia

MOVING REFERENCE FRAME: A Stationary Frame of Reference is when an event is being viewed from an observer who is stationary. It is symbolised by just the symbol of the quantity being referred to: e.g.: t would just be written as t

REFERENCE FRAME: A Reference Frame could be thought of as an observer viewing an event: *see Moving Frame of Reference, Stationary Frame of Reference*.

RELATIVISTIC MASS: The increase in mass of a mass as the velocity increases

STATIONARY REFERENCE FRAME: A Moving Frame is when an event is being viewed from an observer who is moving at a certain velocity. It is symbolised by the symbol of the quantity being referred, followed by an apostrophe: e.g.: t would just be written as t' or t_0.

VELOCITY: A Vector quantity that describes the Rate of change in displacement per unit of time: $\frac{\delta s}{\delta t}$. But when there is a clear context in a positive linear plane in 1-Dimensional Space, it can be expressed as: $\frac{\delta d}{\delta t}$.

TIME: Time is a basic Fundamental SI Unit. It cannot be defined. However, it can be mathematically defined parametrically as a 'fourth dimension'. Time can be thought of as the sequence of intervals in which the events occur in 3-Dimensional Space – just like a Parametrically defined function that is drawn for each interval of t.

APPENDIX of SYMBOLS

ℓ, length

t, time

m, mass

v, velocity

c, speed of light

x, d, distance

F, force

E, energy

ρ, P, momentum

$\rightarrow$, This Equation will be later used in a Substitution

$\Rightarrow$, This Equation is the Final Equation

δ, An infinitisimally low number - as used in Leibniz Notation

For a Physical Quantity, q

q', q_0, As observed in a Stationary Reference Frame

q, As observed in a Moving Reference Frame