$\ldots$

June 12, 2008

Let $]\tau_a, \tau_b[=]-\Gamma/2, \Gamma/2[$ be a finite intervals. For some bounded* real-valued function P of τ, consider the operator

$$-\frac{d^2}{d\tau^2} + P(\tau) \tag{0.1}$$

that acts on functions u with

$$\operatorname{supp}(u) = \overline{\{\tau \in \mathbb{R};\ u(\tau) \neq 0\}} \subset [\tau_a, \tau_b], \tag{0.2}$$

in particular

$$u(\pm\Gamma/2) = 0. \tag{0.3}$$

The ordinary differential equation

$$\left(-\frac{d^2}{d\tau^2} + P(\tau)\right) u(\tau) = \lambda u(\tau), \tag{0.4}$$

with the intial conditions

$$u(-\Gamma/2) = 0 \quad \tfrac{du}{d\tau}(-\Gamma/2) = 1. \tag{0.5}$$

constitutes an initial value problem for which unique solutions exist for every $\lambda \in \mathbb{C}$.

If u additionally satisfies (0.2), then u would be in the domain of the operator and therefore an eigenfunction of (0.1) by definition. The converse can be seen by the following argument. Assume that λ is an eigenvalue but $u(\Gamma/2) \neq 0$. Then there must exist an eigenfunction function v with $v(\pm\Gamma) = 0$. It is easily seen that u can be scaled to fit any initial conditions as above where the derivative does not vanish. Since u and v are linearly independent, v cannot be the solution for any of these intial values. Consequently, v must be the solution to $\frac{du_{P,\lambda}}{d\tau}(-\Gamma/2) = 0$ and we may conclude that in fact $v \equiv 0$ through uniqueness which is a contradiction.

...

*this is assumed so that we may apply general theorems of ordinary differential equations

Differentiating equation (0.4) with respect to λ gives

$$\left(-\frac{d^2}{d\tau^2} + P\right) \frac{du_{P,\lambda}}{d\lambda} = u_{P,\lambda} + \lambda \frac{du_{P,\lambda}}{d\lambda} \tag{0.6}$$

Now consider the function of τ

$$W = u_{P,\lambda} \frac{d^2 u_{P,\lambda}}{d\tau d\lambda} - \frac{du_{P,\lambda}}{d\tau} \frac{du_{P,\lambda}}{d\lambda} \tag{0.7}$$

so that

$$\begin{aligned}
\frac{dW}{d\tau} &= \frac{du_{P,\lambda}}{d\tau} \frac{d^2 u_{P,\lambda}}{d\tau d\lambda} + u_{P,\lambda} \frac{d^3 u_{P,\lambda}}{d\tau^2 d\lambda} - \frac{d^2 u_{P,\lambda}}{d\tau^2} \frac{du_{P,\lambda}}{d\lambda} - \frac{du_{P,\lambda}}{d\tau} \frac{d^2 u_{P,\lambda}}{d\tau d\lambda} && (0.8) \\
&= u_{P,\lambda} \frac{d^3 u_{P,\lambda}}{d\tau^2 d\lambda} - \frac{d^2 u_{P,\lambda}}{d\tau^2} \frac{du_{P,\lambda}}{d\lambda} && (0.9) \\
&= u_{P,\lambda} \left[(P - \lambda) \frac{du_{P,\lambda}}{d\lambda} - u_{P,\lambda} \right] - (P - \lambda) u_{P,\lambda} \frac{du_{P,\lambda}}{d\lambda} && (0.10) \\
&= -u_{P,\lambda}^2. && (0.11)
\end{aligned}$$

From the boundary conditions (0.5) we have

$$u_{P,\lambda}(-\Gamma/2) = 0 = \frac{d}{d\lambda} u_{P,\lambda}(-\Gamma/2), \tag{0.12}$$

hence

$$W(-\Gamma/2) = 0. \tag{0.13}$$

Suppose that $u_{P,\lambda}$ is an eigenfunction so that (0.3) is satisfied. As we have remarked, this can only happen if $\lambda \in \mathbb{R}$. A well-known theorem is applicable now, stating that the eigenfunctions can always be chosen real. In this case, (0.11) states that W is monotonically falling. Now, if we additionally assume in (0.3) that the zero in λ is not simple in, i.e.

$$\frac{du_{P,\lambda}}{d\lambda}(\Gamma/2) = 0 \tag{0.14}$$

then we would have

$$W(\Gamma/2) = 0. \tag{0.15}$$

But this implies that W equals zero on $[-\Gamma/2, \Gamma/2]$ identically, thus $-u_{P,\lambda}^2 \equiv 0$ in (0.11), which contradicts the assumption that $u_{P,\lambda}$ is an eigenfunction.