

$$y(s, t) = v(x(s, t), t)$$

$$\frac{\partial y}{\partial t} = \frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial t}$$

Proof 1:

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= \frac{d^2 v}{dt^2} = \frac{d}{dt} \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} \right) \frac{\partial x}{\partial t} \\ &= \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial x \partial t} \frac{\partial x}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial^2 x}{\partial t^2} + \left(\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 v}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial}{\partial x} \frac{\partial x}{\partial t} \right) \frac{\partial x}{\partial t} \\ &= \frac{\partial^2 v}{\partial t^2} + 2 \frac{\partial^2 v}{\partial x \partial t} \frac{\partial x}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 v}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 \end{aligned}$$

Proof 2:

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= \frac{d^2 v}{dt^2} = \frac{d}{dt} \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} \right) = \frac{d}{dt} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{d}{dt} \frac{\partial x}{\partial t} + \frac{\partial x}{\partial t} \frac{d}{dt} \frac{\partial v}{\partial x} \\ &= \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial x \partial t} \frac{\partial x}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t} \left(\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 v}{\partial x^2} \frac{\partial x}{\partial t} \right) \\ &= \frac{\partial^2 v}{\partial t^2} + 2 \frac{\partial^2 v}{\partial x \partial t} \frac{\partial x}{\partial t} + \frac{\partial v}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 v}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 \end{aligned}$$