

I found a copy of Euler's Analysis of the Infinite. Here's how he did it. Some of his reasoning is going to make you break out in hives:

First get the expansion of e^x with the binomial theorem and defining e in a certain way, so you have:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

and

$$\left(1 + \frac{z}{j}\right)^j = e^z$$

where j is infinitely large.

Then get:

$$(\cos z \pm i \sin z)^n = \cos nz \pm i \sin nz$$

He sort of assumes that but you can prove it with induction.

Then:

$$\cos nz = \frac{(\cos z + i \sin z)^n + (\cos z - i \sin z)^n}{2}$$

and

$$\sin nz = \frac{(\cos z + i \sin z)^n - (\cos z - i \sin z)^n}{2}$$

Then expanding the binomials you have:

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{1 \cdot 2} (\cos z)^{n-2} (\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} (\cos z)^{n-4} (\sin z)^4 - \dots (*)$$

$$\sin nz = \frac{n}{1} (\cos z)^{n-1} (\sin z) - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (\cos z)^{n-3} (\sin z)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\cos z)^{n-5} (\sin z)^5 - \dots$$

Then (I quote) "134. Let the arc z be infinitely small, then $\sin z = z$ and $\cos z = 1$. If n is an infinitely large number, so that nz is a finite number, say $nz = v$, then, since $\sin z = z = \frac{v}{n}$ we have

$$\cos v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \dots$$

and

$$\sin v = v - \frac{v^3}{3!} + \frac{v^5}{5!} - \dots"$$

[Then proceeds some matter where he calculates the first 30 or so terms of each expression to 27 place of accuracy...had some time on his hands I guess]. Then he proceeds to derive the complex exponential-trigonometric identity:

"138. Once again we use the formulas in 133. [(*)] where we let z be an infinitely small arc and let n be an infinitely large number j , so that jz has a finite value v . Now we have $nz = v$ and $z = \frac{v}{j}$ so that $\sin z = \frac{v}{j}$ and $\cos z = 1$. With these substitutions,

$$\cos v = \frac{(1 + \frac{iv}{j})^j + (1 - \frac{iv}{j})^j}{2}$$

and

$$\sin v = \frac{(1 + \frac{iv}{j})^j - (1 - \frac{iv}{j})^j}{2i}$$

. In the preceding chapter we saw that $(1 + \frac{z}{j})^j = e^z$ where e is the base of the natural logarithms. When we let $z = iv$ and the we let $z = -iv$ we obtain $\cos v = \frac{e^{iv} + e^{-iv}}{2}$ and $\sin v = \frac{e^{iv} - e^{-iv}}{2i}$. From these equations we understand how complex exponentials can be expressed by real sines and cosines, since $e^{iv} = \cos v + i \sin v$ and $e^{-iv} = \cos v - i \sin v$."