Notes on Differential Equations

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Part 1

Ordinary Differential Equations

CHAPTER 1

The Basics of Ordinary Differential Equations

The subject of differential equations is important in the majority of the STEM fields, primarily in engineering and mathematics. Some models in biology and chemistry use differential equations as well. Our first goal is to understand what differential equations are; thus, the first section in this chapter will define what we mean by a differential equation and how we classify the different types of differential equations. Next, we will start with the most simple examples of differential equations and what is required to solve them; take note that we will focus only on first order equations in this chapter. Following this, we will discuss the method of separation of variables, how to solve linear differential equations, and how to solve exact differential equations. To understand the method of solving exact differential equations, you should at least know how to take partial derivatives of multi-variable functions that live in $\mathbb{R}^3$.

1. What Is A Differential Equation?

What is a differential equation? The adjective "differential" seems to suggest that we will be dealing with derivatives...but derivatives of what? It turns out that a differential equation relates an unknown function with its derivatives. For a more formal definition, let us consider the following:

Definition 1.1. Let f be a function. Let $x : \mathbb{R} \to \mathbb{R}$ be an unknown differentiable function in variable $t \in \mathbb{R}$, and let $x'(t), x''(t), \ldots, x^{(n)}(t)$ denote the derivatives of $x(t)$. Then the equation

$$f(t, x, x', x'', \ldots, x^{(n)}) = 0 \tag{1.1}$$

is called a **ordinary differential equation** (which we will abbreviate as ODE). We define the **order** of a differential equation to be the order of the highest derivative it contains.

Definition 1.2. Let $u : \mathbb{R}^n \to \mathbb{R}$ be an unknown differentiable function of the multiple independent variables $x_1, \ldots, x_n$. Then the equation

$$F\left(u, x_1, \ldots, x_n, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \ldots\right) = 0 \tag{1.2}$$

is called a **partial differential equation** (which we will abbreviate as PDE). Again, the order of the partial differential equation is the order of the highest derivative appearing in the partial differential equation.

By the above definition, (1.1) would be an n-th order ODE since $x^{(n)}(t)$ is the highest order derivative of $x(t)$. Likewise, (1.2) would be an n-th order PDE. ODEs and PDEs are the two most common types of differential equations. Both types of differential equations are useful in their own ways, but **we will focus only on ODEs for the time being** (the first part of this book will be on ODEs only; part 2 will cover topics in PDEs).

Within each of these main classes, we can further categorize a differential equation by determining whether or not an ODE or PDE is **linear** or **non-linear**. An ODE is said to be linear if the function f is a linear function in $x(t)$ and its derivatives. A function $f : \mathbb{R} \to \mathbb{R}$ is linear if for all $c, x, y \in \mathbb{R}$, it is true that $f(x+y) = f(x) + f(y)$ and $f(cx) = cf(x)$. A linear ODE can be written as

$$a_n(t)\, x^{(n)} + a_{n-1}(t)\, x^{(n-1)} + \cdots + a_1(t)\, x' + a_0(t)\, x = g(t). \tag{1.3}$$

A PDE is linear if the function F is a linear function of u and all its derivatives. Otherwise, the ODE or PDE is said to be non-linear. To get a better understanding of what this means, let's look at a few examples.

Example 1.1 (Harmonic Oscillator). The differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{1.4}$$

is a second order linear ODE. This models the motion of a spring mass system without any damping factors.

Example 1.2 (Two-dimensional Wave Equation). The differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{1.5}$$

is a second order linear PDE. This models the vibration of a membrane with fixed boundary.

Example 1.3 (Navier-Stokes Equations). The differential equation

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 + \mathbf{f} \tag{1.6}$$

is a second order non-linear PDE. It's second order due to the fact that

$$\nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

in $\mathbb{R}^n$. This is the Navier-Stokes equation for an incompressible fluid, and is the subject of one of the seven Millennium Prize Problems proposed by the Clay Mathematics Institute.

Example 1.4 (Motion of a Pendulum). The differential equation

$$\frac{d^2u}{dt^2} + \frac{g}{L} \sin(u) = 0 \tag{1.7}$$

is a second order non-linear ODE. This models the motion of a pendulum with length L.

Now that we have a better understanding for what a differential equation is, we should now discuss a special type of ODE or PDE – the **initial value problem**.

Definition 1.3 (Initial Value Problem). An **initial value problem** (abbreviated as IVP) is a differential equation coupled with an **initial condition** $f(x_0) = y_0$, where f is the solution of our ODE (or PDE) with $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}$.

The thing that makes an IVP different from the general ODE or PDE is that a solution of the IVP is said to be a **particular solution** (i.e., the solution contains no arbitrary constants that may arise from techniques needed to solve the equation). A solution of an ODE or PDE (given that one exists to begin with) with no initial conditions is called a **general solution**. For example, if we have the differential equation

$$\frac{dy}{dx} + y = x, \tag{1.8}$$

it's general solution is $y(x) = x - 1 + Ce^{-x}$. However, if we couple it with the initial condition $y(0) = 0$, we then end up with the particular solution $y(x) = x - 1 + e^{-x}$. For various initial conditions, however, the solution to an ODE or PDE may or may not exist! This leads into the next important topic – existence and uniqueness of solutions to differential equations. For now, we will restrict ourselves to the case of first order ODEs. We will state the theorem on existence and uniqueness of solutions, but will continue the discussion of existence and uniqueness in the next section, once we have a means for solving simple ODEs.

Theorem 1.1 (Existence and Uniqueness of Solutions). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function with partial derivative f_y. Suppose that both $f(x, y)$ and $f_y(x, y)$ are continuous on some rectangle $R \subset \mathbb{R}^2$ such that for some $(a, b) \in \mathbb{R}^2$, we have $(a, b) \in R$. Then, for some open interval I containing the point a, the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b \tag{1.9}$$

has exactly one solution that is defined on the interval I.

We should now have a much better understanding for what differential equations are and what their various characteristics are. We will now start the long journey of understanding how to solve different kinds of differential equations. I hope you're ready to march forward with me!

2. ODEs of the form $\frac{dy}{dx} = f(x)$.

In the previous section, we learned about what differential equations were. In this section, we first focus on the most basic of ODEs. First, let us consider differential equations of the form

$$\frac{dy}{dx} = f(x, y). \tag{2.1}$$

In the scenario where $f(x, y)$ isn't dependent on the variable y, we are left with the much simpler differential equation

$$\frac{dy}{dx} = f(x). \tag{2.2}$$

The solution to this differential equation is found by integrating both sides of (2.2). The general solution would be

$$y(x) = \int f(x)\, dx + C \tag{2.3}$$

and thus would define a **one parameter family of solutions** dependent on C. If (2.2) was coupled with the initial condition $y(x_0) = y_0$ and we defined $G(x) \equiv \int f(x)\, dx$, then the general solution would become the particular solution

$$y(x) = G(x) + y_0 - G(x_0). \tag{2.4}$$

As we solve any ODE, we will always end up finding the general solution first. If the problem is an IVP, we then couple the general solution we found with the initial condition to find the particular solution. We now consider the following examples.

Example 2.1.

Example 2.2.

3. Separation of Variables

4. First Order Linear ODEs

5. Exact First Order ODEs

6. Exercises

CHAPTER 2

Second and Higher Order ODEs

CHAPTER 3

Numerical Methods in ODEs

CHAPTER 4

Laplace Transforms

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Power Series Methods in ODEs

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Systems of ODEs and Matrix Methods

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Dynamical Systems with an Introduction to Chaos

CHAPTER 8

A More Theoretical Approach to ODEs

Part 2

Partial Differential Equations