$$[\lambda] = \tfrac{3}{2} = [\bar{\lambda}] \tag{3.22d}$$

$$[D] = 2. \tag{3.22e}$$

So, as noted in Chapter 1, the highest-dimension field (D) in the supermultiplet must transform as a total divergence. Further, since $\partial_\mu \partial^\mu C$ also transforms as a total divergence, we see that $D - \frac{1}{2}\partial_\mu \partial^\mu C$ does too, and so therefore does the entire coefficient of $\theta\theta\bar{\theta}\bar{\theta}$ in $V(x, \theta, \bar{\theta})$. This is the justification of the claim in §2.5 that the variation of the D-term in a vector superfield is a total divergence.

Since the only requirement (3.2) for a vector superfield is that it be real, it is easy to construct a particular example of one using the chiral superfield Φ and the anti-chiral superfield $\Phi^\dagger$ given in (2.27), (2.28). For instance

$$\begin{aligned} \mathrm{i}(\Phi - \Phi^\dagger) = {} & \mathrm{i}(\varphi - \varphi^\dagger) + \mathrm{i}\sqrt{2}(\theta\psi - \bar{\theta}\bar{\psi}) + \mathrm{i}\,\theta\theta F - \mathrm{i}\,\bar{\theta}\bar{\theta}F^\dagger \\ & - \theta\sigma^\mu\bar{\theta}\,\partial_\mu(\varphi + \varphi^\dagger) - \frac{1}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\,\partial_\mu\psi + \frac{1}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\,\partial_\mu\bar{\psi} \\ & - \tfrac{1}{4}\mathrm{i}\,\theta\theta\bar{\theta}\bar{\theta}\,\partial_\mu\partial^\mu(\varphi - \varphi^\dagger) \end{aligned} \tag{3.23}$$

has the form (3.4) with

$$C = \mathrm{i}(\varphi - \varphi^\dagger) \tag{3.24a}$$

$$\chi = \sqrt{2}\,\psi \tag{3.24b}$$

$$\tfrac{1}{2}(M + \mathrm{i}\,N) = F \tag{3.24c}$$

$$V_\mu = -\,\partial_\mu(\varphi + \varphi^\dagger) \tag{3.24d}$$

$$\lambda = 0 \tag{3.24e}$$

$$D = 0. \tag{3.24f}$$

Of course, for this identification to work the dimension of the fields φ, ψ, F must be shifted by one unit from the canonical dimensions (1.175), (1.179), (1.194) which they are assigned in order to make the usual identification with quarks, leptons etc. Nevertheless the force of the observation (3.24) becomes clear when we note that the vector potential V_μ for the superfield $\mathrm{i}(\Phi - \Phi^\dagger)$ is a pure U(1) gauge transformation, and this suggests how to make a supersymmetric generalization of gauge invariance.

3.3 Supersymmetric gauge invariance

We start with the familiar local U(1) gauge invariance (of QED). Under such a gauge transformation the vector potential transforms as

$$V_\mu(x) \rightarrow V_\mu{}'(x) = V_\mu(x) + \partial_\mu\Lambda(x) \tag{3.25}$$

where $\Lambda(x)$ is a 'gauge function'. The discussion at the end of §3.2 suggests an immediate way to supersymmetrize the transformation (3.25). Since V_μ is a component of the vector superfield (3.4), and $\partial_\mu\Lambda \equiv \partial_\mu(\varphi + \varphi^\dagger)$ is in $\mathrm{i}(\Phi - \Phi^\dagger)$, Wess and Zumino[1] suggested that the superfield transforms as

$$V(x, \theta, \bar{\theta}) \to V'(x, \theta, \bar{\theta}) = V(x, \theta, \bar{\theta}) + \mathrm{i}[\Phi(x, \theta, \bar{\theta}) - \Phi^\dagger(x, \theta, \bar{\theta})] \quad (3.26)$$

under a U(1) gauge transformation. In fact, it is clear from (3.24) that in a gauge theory the fields C, χ, M, N are not physical degrees of freedom, since they can be 'gauged away' by a suitable choice of $\varphi - \varphi^\dagger$, ψ, F while still leaving $\Lambda = \varphi + \varphi^\dagger$ arbitrary. Then in the 'Wess–Zumino gauge' the vector superfield is

$$V_{\mathrm{WZ}}(x, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta}V_\mu(x) + \mathrm{i}\,\theta\theta\bar{\theta}\bar{\lambda}(x) - \mathrm{i}\,\bar{\theta}\bar{\theta}\theta\lambda(x) + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \quad (3.27)$$

and from (3.24) the fields λ, $\bar{\lambda}$, D are gauge invariant while V_μ transforms as in (3.25). Note that in the Wess–Zumino gauge the field D, which from (3.18) transforms as a total divergence, *is* the coefficient of $\theta\theta\bar{\theta}\bar{\theta}$. Also all powers V^n_{WZ} with $n > 2$ vanish, since they will involve at least θ^3.

The only non-zero power is

$$V^2_{\mathrm{WZ}}(x, \theta, \bar{\theta}) = -(\theta\sigma^\mu\bar{\theta})(\bar{\theta}\bar{\sigma}^\nu\theta)V_\mu V_\nu = \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}V^\mu V_\mu \quad (3.28)$$

using (1.74*a*) and (A7). Such a term supplies a mass for the vector field, and thereby breaks the gauge invariance. Since the massive vector theory is not gauge invariant, the degrees of freedom C, χ, M, N *are* physical and cannot be gauged away. In fact, as is clear from their dimensionality (3.22), the field C supplies the longitudinal mode of the vector field, while χ, $\bar{\chi}$ supply the extra degrees of freedom for the massive gaugino field.

To construct a supersymmetric gauge field theory we need first to construct the field strength superfield, and secondly to couple the vector superfield to the charged (chiral) matter superfield in a gauge-invariant way. We have already observed that the fields λ, $\bar{\lambda}$, $V_{\mu\nu}$ and D form an irreducible representation of the supersymmetry algebra, and that all of these fields are gauge invariant. This suggests that the field strength superfield is a spinor (chiral) superfield, since the lowest-dimension field is λ_α with $[\lambda_\alpha] = \frac{3}{2} = [\bar{\lambda}]$ while $[V_{\mu\nu}] = 2 = [D]$. It is easy to construct the required superfield W_α using covariant derivatives. Let

$$W_\alpha \equiv \bar{D}^2 D_\alpha V. \quad (3.29)$$

Then from (3.7)

$$W_\alpha| = 4\mathrm{i}\,\lambda_\alpha \quad (3.30)$$

and we see that the lowest-dimension field is λ, as required. Also, it follows from (3.29) that

$$\bar{D}_{\dot{\beta}}W_\alpha = 0 \quad (3.31)$$

since

$$\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}}\bar{D}_{\dot{\delta}} = 0 \tag{3.32}$$

automatically. Thus W_α *is* a chiral superfield satisfying the constraint (2.20), which means that it has the general form

$$W_\alpha(y, \theta) = 4\mathrm{i}\,\lambda_\alpha(y) + \theta^\beta \varphi_{\alpha\beta}(y) + \theta\theta F_\alpha(y) \tag{3.33a}$$

as in (2.25), with

$$y^\mu = x^\mu + \mathrm{i}\,\theta\sigma^\mu\bar{\theta} \tag{3.33b}$$

but now $\varphi_{\alpha\beta}$ is a bosonic field and F_α a spinor field. It follows from (3.32), (3.28) and (3.11) that

$$D_\beta W_\alpha| = \varphi_{\alpha\beta} = D_\beta \bar{D}^2 D_\alpha V| = \epsilon_{\beta\gamma}[4\delta^\gamma_\alpha D + 2\mathrm{i}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\gamma V_{\mu\nu}]. \tag{3.34}$$

Also, using (2.17) and (3.7)

$$\begin{aligned} D^2 W_\alpha| &= -4F_\alpha = D^2\bar{D}^2 D_\alpha V| = D^2[\bar{D}^2, D_\alpha]V| \\ &= D^2(\bar{D}_{\dot{\beta}}\{D_\alpha, \bar{D}^{\dot{\beta}}\} - \{D_\alpha, \bar{D}_{\dot{\beta}}\}\bar{D}^{\dot{\beta}})V| \\ &= -\,4\mathrm{i}\,\sigma^\mu_{\alpha\dot{\beta}}\,\partial_\mu D^2\bar{D}^{\dot{\beta}}V| = -\,16\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\bar{\lambda}^{\dot{\alpha}}. \end{aligned} \tag{3.35}$$

So substituting into (3.33*a*) gives the field strength superfield

$$\begin{aligned} W_\alpha(y, \theta) = 4\mathrm{i}\,\lambda_\alpha(y) &- [4\delta_\alpha{}^\beta D(y) + 2\mathrm{i}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta V_{\mu\nu}(y)]\theta_\beta \\ &+ 4\theta^2\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\bar{\lambda}^{\dot{\alpha}} \end{aligned} \tag{3.36}$$

with y given by (3.33*b*). To construct the (gauge-invariant) supersymmetric pure gauge theory we want the F-component of $W^\alpha W_\alpha$, since, as shown in Chapter 2, this transforms as a total divergence under supersymmetry transformations and therefore yields an invariant action. A simple calculation yields

$$\tfrac{1}{32}(W^\alpha W_\alpha)_F = -\,\tfrac{1}{4}V^{\mu\nu}V_{\mu\nu} + \mathrm{i}\,\lambda\sigma^\mu\,\partial_\mu\bar{\lambda} - \tfrac{1}{4}V^{\mu\nu}({}^*V_{\mu\nu}) + \tfrac{1}{2}D^2 \tag{3.37}$$

where

$${}^*V_{\mu\nu} \equiv \frac{\mathrm{i}}{2}\,\epsilon_{\mu\nu\rho\sigma}V^{\rho\sigma} \tag{3.38}$$

is the dual field strength tensor. We can use (3.37) as the supersymmetric generalization of the familiar kinetic terms $-\frac{1}{4}V_{\mu\nu}V^{\mu\nu}$ of the U(1) gauge field, since the term involving ${}^*V_{\mu\nu}$ is a total divergence and so does not affect the equations of motion. The D-field is an auxiliary field which can be eliminated using the equations of motion. The gaugino contribution can be rewritten in terms of the (four-component) Majorana spinor