ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Maths of Waves and Fields

16th January 2008, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

---

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

P.T.O.

1. (a) The function $f(x)$ is defined on $-\infty < x < \infty$ by

$$f(x) = \begin{cases} e^{-\alpha x} & \text{for} \quad 0 < x < \infty, \\ 0 & \text{for} \quad -\infty < x < 0, \end{cases}$$

where α is a positive constant. Find its Fourier transform $g(k)$ and sketch $|g(k)|^2$.
[9 marks]

(b) Chebyshev polynomials satisfy the differential equation

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + Ny = 0,$$

for certain values of the constant N. One such polynomial has the form

$$y(x) = a_0 + a_2 x^2.$$

Find the value of N for which this is a solution to the equation, and express a_2 in terms of a_0.
[8 marks]

(c) The function $f(x) = 3x - 5x^3$ can be expanded in Legendre polynomials $P_l(x)$ as

$$f(x) = \sum_{l=1}^{3} c_l P_l(x).$$

Given that $P_1(x) = x$, use the orthogonality of the Legendre polynomials on $-1 \leq x \leq +1$ to find the coefficient c_1 of $P_1(x)$ in this series.
[8 marks]

P.T.O.

2. (a) The function $f(x)$ is defined on the region $0 \leq x \leq \pi$ by

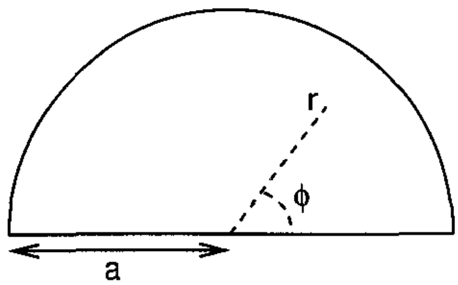
$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \pi/4, \\ 0 & \text{for} \quad \pi/4 < x < \pi. \end{cases}$$

It can be represented by the Fourier sine series

$$f(x) = \sum_{m=1}^{\infty} b_m \sin(mx).$$

Find its Fourier coefficients b_m. [7 marks]

(b) A semicircular region of radius a is shown in the diagram.



The electrostatic potential $V(r, \phi)$ inside the semicircle satisfies Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0,$$

where (r, ϕ) are plane polar coordinates. The potential along the straight side of the region is held at zero,

$$V(r, 0) = V(r, \pi) = 0.$$

Verify that the separable function

$$V(r, \phi) = r^m \sin(m\phi)$$

satisfies Laplace's equation. Use the boundary conditions on the ϕ dependence to determine the allowed values of m. Explain also why negative values for m are not allowed. Hence write down the general expression for the potential inside the semicircle. [13 marks]

(c) The potential on the curved side is given by

$$V(a, \phi) = \begin{cases} V_0 & \text{for} \quad 0 < \phi < \pi/4, \\ 0 & \text{for} \quad \pi/4 < \phi < \pi. \end{cases}$$

Using your result for part (a) find, in series form, the potential $V(r, \phi)$ inside the semicircle. [5 marks]

P.T.O.

3. The axis of a long, thin-walled cylindrical tube of radius a lies along the z-axis. The transverse displacement $f(z, \phi, t)$ of the wall of this tube satisfies the wave equation in cylindrical polar coordinates,

$$\frac{\partial^2 f}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2 f}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2},$$

where c is a constant.

(a) By substituting a separable function of the form

$$f(z, \phi, t) = \Phi(\phi)\mathrm{e}^{\mathrm{i}(kz+\omega t)}$$

into the wave equation, show that $\Phi(\phi)$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -m^2\Phi,$$

and find an equation relating the angular frequency ω to the constants k and m. [7 marks]

(b) What boundary condition must $\Phi(\phi)$ satisfy? By solving the differential equation for $\Phi(\phi)$ subject to this condition, find the allowed eigenvalues m^2 and the corresponding eigenfunctions of ϕ. [5 marks]

(c) Assuming that k and ω are both positive, what is the direction of propagation of these waves? Find their phase and group velocities. How does the propagation of waves with $m = 0$ differ from those with $m \neq 0$? [8 marks]

(d) If angular frequency ω is in the range

$$\frac{c}{a} < \omega < \frac{2c}{a},$$

show that three independent modes of vibration can propagate along the tube. [5 marks]

P.T.O.

4. The temperature distribution $\phi(x,t)$ inside a uniform metal bar of length L satisfies the heat-flow equation

$$\frac{1}{D}\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2},$$

where D is a constant. The end of the bar at $x = 0$ is insulated and the end at $x = L$ is held at zero temperature. The corresponding boundary conditions on $\phi(x,t)$ are

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = 0, \qquad \phi(L,t) = 0.$$

(a) By trying a separable solution of the form $\phi(x,t) = X(x)T(t)$, find the eigenvalue problem (differential equation and boundary conditions) satisfied by $X(x)$. Find also the differential equation for $T(t)$. [7 marks]

(b) Show that the eigenfunctions $X_n(x)$ have the form

$$X_n(x) = \cos(k_n x),$$

and find the allowed values of the constants k_n. [6 marks]

(c) Show that the eigenfunctions are orthogonal on the interval $0 \leq x \leq L$. [6 marks]

(d) Show that the general solution for $\phi(x,t)$ is of the form

$$\phi(x,t) = \sum_n A_n \cos(k_n x)\, \mathrm{e}^{-\gamma_n t},$$

and give an expression for the relaxation rates γ_n. Sketch the shape of the temperature distribution as a function of x at large time, $t > 1/\gamma_1$. (You may assume that all of the A_n are nonzero.) [6 marks]

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

SYMBOL	DESCRIPTION	NUMERICAL VALUE
c	Velocity of light in vacuum	$299\,792\,458$ m s^{-1}, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}$ N A^{-2}, exactly
ϵ_0	Permittivity of vacuum where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$	8.854×10^{-12} C^2 N^{-1} m^{-2}
h	Planck constant	6.626×10^{-34} J s
$\hbar$	$h/2\pi$	1.055×10^{-34} J s
G	Gravitational constant	6.674×10^{-11} m^3 kg^{-1} s^{-2}
e	Elementary charge	1.602×10^{-19} C
eV	Electronvolt	1.602×10^{-19} J
α	Fine-structure constant, $\frac{e^2}{4\pi\epsilon_0 \hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	9.109×10^{-31} kg
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	9.274×10^{-24} J T^{-1}
R_∞	Rydberg energy $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha}\frac{\hbar}{m_e c}$	0.5292×10^{-10} m
Å	Angstrom	10^{-10} m
m_p	Proton mass	1.673×10^{-27} kg
$m_p c^2$	Proton rest-mass energy	938.272 MeV
$m_n c^2$	Neutron rest-mass energy	939.566 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	5.051×10^{-27} J T^{-1}
fm	Femtometre or fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12} m(^{12}\mathrm{C}$ atom)	1.661×10^{-27} kg
N_A	Avogadro constant, atoms in gram mol	6.022×10^{23} mol^{-1}
T_t	Triple-point temperature	273.16 K
k	Boltzmann constant	1.381×10^{-23} J K^{-1}
R	Molar gas constant, $N_A k$	8.315 J mol^{-1} K^{-1}
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60}\frac{k^4}{\hbar^3 c^2}$	5.670×10^{-8} W m^{-2} K^{-4}
M_E	Mass of Earth	5.97×10^{24} kg
R_E	Mean radius of Earth	6.4×10^6 m
g	Standard acceleration of gravity	$9.806\,65$ m s^{-2}, exactly
atm	Standard atmosphere	101 325 Pa, exactly
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.961×10^8 m
$L_\odot$	Solar luminosity	3.846×10^{26} W
$T_\odot$	Solar effective temperature	5800 K
AU	Astronomical unit, mean Earth-Sun distance	1.496×10^{11} m
pc	Parsec	3.086×10^{16} m
	Year	3.156×10^7 s