

A Problem in Fluid Dynamics

Matthew Lundy and Aleksei Beltukov

October 7, 2008

Consider a cylindrical tank filled to a height H with an ideal, incompressible fluid initially at rest. At time zero, a rotationally symmetric circular hole of finite area is made in the bottom of the tank—how does the tank drain?

On one hand we have Euler's equations of motion for an ideal, incompressible fluid. They are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = g \hat{\mathbf{k}} - \frac{\nabla p}{\rho} \quad (1)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2)$$

where \mathbf{v} is the Eulerian description of the velocity of the fluid, $p = p(x, y, z, t)$ is the pressure, and g and ρ are the acceleration due to gravity and the fluid density, respectively.

There are also two free surfaces associated with this problem: the free-surface at the top of the fluid and the jet emerging from the hole in the bottom of the tank. These surfaces are material surfaces, so if we denote the surfaces as $f_i(x, y, z, t) = 0$ for $i = 1, 2$ we can write

$$\frac{\partial f_i}{\partial t} + (\mathbf{v} \cdot \nabla) f_i = 0 \quad \text{for } i=1,2.$$

We also have the slip condition for the boundaries of the tank, written as

$$\frac{\partial \mathbf{v}}{\partial \nu} = 0$$

which is to say that along the boundaries of the tank, the velocity is entirely tangential to the boundary.

However, we note that on the free surfaces of the fluid, there is an unbalanced flux, which means that

$$\nabla \cdot \mathbf{v} \neq 0.$$

At the moment, we cannot reconcile this difficulty.

On the other hand, Helmholtz tells that any vector field that is initially irrotational will remain irrotational for all time when subject to a conservative field. If the flow is forever irrotational, then there should exist a velocity potential Φ such that

$$\mathbf{v} = \nabla \Phi.$$

Also, even though the boundaries of the fluid have non-zero divergence, the interior of the fluid must remain divergenceless because of incompressibility. This allows us to write

$$\begin{aligned}\nabla \cdot \mathbf{v} &= 0 \\ \nabla \cdot \nabla \Phi &= 0 \\ \Delta \Phi &= 0\end{aligned}$$

and the problem should reduce to solving Laplace's equation at each instant of time subject to certain boundary conditions. But what are the boundary conditions? What's more, what are the initial conditions? If Φ is initially zero, it will remain zero for all time. And where does gravity come in?

The problem is actually a much more general one. How do you model unsteady fluid flow subject to a gravitational field?