

Consider a two-sheeted hyperboloid defined in Cartesian  $\mathbb{R}^3$  coordinates by the equation  $x^2 - y^2 - z^2 = K^2$ . Each sheet of the hyperloid “cups” out along the  $x$ -axis. Now imagine that we want to determine the volume contained in both sheets of the two-sheeted hyperboloid between the space from  $-J$  to  $J$  the the  $x$ -axis (where  $J \geq K$ ).

Using rectangular coordinates, the combined volume,  $V$ , of the two sheets can be computed by

$$V = 2 \int_K^J \int_{-\sqrt{x^2-K^2}}^{\sqrt{x^2-K^2}} \int_{-\sqrt{x^2-y^2-K^2}}^{\sqrt{x^2-y^2-K^2}} dz dy dx.$$

Performing the first integration with respect to  $dz$ , we get

$$V = 2 \int_K^J \int_{-\sqrt{x^2-K^2}}^{\sqrt{x^2-K^2}} 2\sqrt{x^2-y^2-K^2} dy dx$$

Performing the second integration with respect to  $dy$ , we get

$$V = 2 \int_K^J 2 \cdot \frac{1}{2} (y\sqrt{x^2-y^2-K^2} + (x^2-K^2)\arcsin(\frac{y}{\sqrt{x^2-K^2}})) \Big|_{-\sqrt{x^2-K^2}}^{+\sqrt{x^2-K^2}} dx$$

Evaluating the limits for  $y$ ,

$$\text{observe that } (y\sqrt{x^2-y^2-K^2}) \Big|_{-\sqrt{x^2-K^2}}^{+\sqrt{x^2-K^2}} = 0.$$

Also evaluating the limits for  $y$ ,

$$\text{observe that } (x^2-K^2)\arcsin(\frac{y}{\sqrt{x^2-K^2}}) \Big|_{-\sqrt{x^2-K^2}}^{+\sqrt{x^2-K^2}} = \pi(x^2-K^2)$$

because  $\arcsin(\frac{+\sqrt{x^2-K^2}}{\sqrt{x^2-K^2}}) = \arcsin(1) = \frac{\pi}{2}$  and

$$\arcsin(\frac{-\sqrt{x^2-K^2}}{\sqrt{x^2-K^2}}) = \arcsin(-1) = -\frac{\pi}{2},$$

$$\text{so } \arcsin(\frac{+\sqrt{x^2-K^2}}{\sqrt{x^2-K^2}}) - \arcsin(\frac{-\sqrt{x^2-K^2}}{\sqrt{x^2-K^2}}) = \pi$$

Performing the third integration with respect to  $dx$ , we get

$$V = 2\pi(x^3/3 - K^2x) \Big|_K^J$$

$$\text{so } V = \frac{2\pi}{3}J^3 - 2\pi K^2J - \frac{2\pi}{3}K^3 + 2\pi K^3$$

$$\text{so } V = \frac{2\pi}{3}(J^3 - K^3) + 2\pi(K^3 - K^2J)$$

Now let us change variables in the volume integrals from  $\mathbb{R}^3$  Cartesian  $x, y, z$  rectangular coordinates to  $\mathbb{R}_{1,2}^3$  pseudo-spherical  $\rho, \chi, \theta$  coordinates.

In psuedo-spherical coordinates,  $\rho$  will represent the directed distance of a given point from the origin (in this case  $\rho = K$ ),  $\chi$  will represent 'latitude' from 0 to  $\cosh^{-1}(\frac{J}{K})$ , and  $\theta$  will represent 'longitude' from 0 to  $2\pi$  radians.

The parametric equations describing the relation between the pseudo-spherical and rectangular coordinates are given as follows:

$$x = \rho \cosh(\chi),$$

$$y = \rho \sinh(\chi) \cos(\theta),$$

$$z = \rho \sinh(\chi) \sin(\theta),$$

To transform integrals from rectangular to pseudo-spherical coordinates,

we must compute the Jacobian of the transformation,  $\det \left( \frac{\partial(x,y,z)}{\partial(\rho,\chi,\theta)} \right)$ . Since

$$x = \rho \cosh(\chi),$$

$$y = \rho \sinh(\chi) \cos(\theta),$$

$z = \rho \sinh(\chi) \sin(\theta)$ , and remembering the following derivatives:

$(\sinh(\chi))' = \cosh(\chi)$ ,  $(\cosh(\chi))' = \sinh(\chi)$ , we have

$$\frac{\partial(x,y,z)}{\partial(\rho,\chi,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \chi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \chi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \chi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cosh(\chi) & \rho \sinh(\chi) & 0 \\ \sinh(\chi) \cos(\theta) & \rho \cosh(\chi) \cos(\theta) & -\rho \sinh(\chi) \sin(\theta) \\ \sinh(\chi) \sin(\theta) & \rho \cosh(\chi) \sin(\theta) & \rho \sinh(\chi) \cos(\theta) \end{pmatrix}$$

Thus  $\det \left( \frac{\partial(x,y,z)}{\partial(\rho,\chi,\theta)} \right) = \rho^2 \sinh(\chi) \cosh^2(\chi) \cos^2(\theta) - \rho^2 \sinh^3(\chi) \sin(\theta) + \rho^2 \sinh(\chi) \cosh^2(\chi) \sin^2(\theta) - \rho^2 \sinh^3(\chi) \cosh^2(\chi) \sin^2(\theta)$

So  $\det \left( \frac{\partial(x,y,z)}{\partial(\rho,\chi,\theta)} \right) = \rho^2 \sinh(\chi) \cos^2(\theta) (\cosh^2(\chi) - \sinh^2(\chi)) + \rho^2 \sinh(\chi) \sin^2(\theta) (\cosh^2(\chi) - \sinh^2(\chi))$

So  $\det \left( \frac{\partial(x,y,z)}{\partial(\rho,\chi,\theta)} \right) = \rho^2 \sinh(\chi) (\cos^2(\theta) + \sin^2(\theta)) = \rho^2 \sinh(\chi)$

Consequently the transformation of the triple integral from rectangular to pseudo-spherical coordinates takes the form

$$V = \int_{-K}^K \int_0^{\cosh^{-1}(\frac{J}{K})} \int_0^{2\pi} \rho^2 \sinh(\chi) d\theta d\chi d\rho.$$

Performing the first integration with respect to  $d\theta$ , we get

$$V = 2\pi \int_{-K}^K \int_0^{\cosh^{-1}(\frac{J}{K})} \rho^2 \sinh(\chi) d\chi d\rho$$

Performing the second integration with respect to  $d\chi$ , we get

$$V = 2\pi \int_{-K}^K \rho^2 \cosh(\chi) d\rho \Big|_0^{\cosh^{-1}(\frac{J}{K})} = 2\pi \int_{-K}^K \rho^2 \left( \frac{J}{K} - 1 \right) d\rho$$

Performing the third integration with respect to  $d\rho$ , we get

$$V = 2\pi \int_{-K}^K \rho^2 \left( \frac{J}{K} - 1 \right) d\rho = \frac{2\pi}{3} \rho^3 \left( \frac{J}{K} \right) - \frac{2\pi}{3} \rho^3 \Big|_{-K}^K$$

$$\text{so } V = \frac{4\pi}{3} (K^2 J - K^3)$$

The obvious question is why don't the answers from each method agree?