The following is from the first chapter of Merzbacher (2nd edition)

> In the "old quantum theory" the classical phase (or action) integrals for a conditionally periodic motion were required to be quantized according to
>
> $$\oint p_i dq_i = n_i h \tag{1}$$
>
> where the *quantum numbers* n_i are integers, and each integral is taken over the full period of the generalized coordinate q_i.
>
> **Exercise** Calculate the quantized energy levels of a linear harmonic oscillator of angular frequency ω in the old quantum theory.

Since the last post, I was able to arrive at the answer which is think is correct (I was having problems with generalized coordinates and *integration!*. Here is my solution.

The Lagrangian for the LHO is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{2}$$

The solution to the LHO equations of motion is:

$$x(t) = x_0 cos(\omega t) \tag{3}$$

where $\omega^2 = k/m$.

Choosing $\theta = \omega t$ as the generalized coordinate, then $\dot{\theta} = \omega$ and the equations of motion are

$$\begin{aligned} x(\theta) &= x_0 \cos(\theta) \qquad (4)\\ \dot{x}(\theta) &= -x_0 \dot{\theta} \sin(\theta) \qquad (5) \end{aligned}$$

So that,

$$L = \frac{1}{2}mx_0^2\dot{\theta}^2 \sin(\theta)^2 - \frac{1}{2}kx_0^2 \cos\theta^2$$

and

$$H = \frac{1}{2}mx_0^2\dot{\theta}^2 \sin^2\theta + \frac{1}{2}kx_0^2 \cos^2\theta$$

Substituting in for k yields,

$$H = \frac{1}{2}mx_0^2\omega^2 \sin^2\theta + \frac{1}{2}mx_0^2\omega^2 \cos^2\theta = \frac{1}{2}mx_0^2\omega^2 \tag{6}$$

We obtain P from

$$P = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} = m x_0^2 \dot{\theta} \sin(\theta)^2$$

So that (1) becomes

$$\begin{aligned} \oint p_i dq_i &= m x_0^2 \dot{\theta} \int_0^{2\pi} \sin(\theta)^2 d\theta = n_i h \\ &= m x_0^2 \omega \pi = n_i h \end{aligned}$$

Or,

$$\omega = \frac{2n\hbar}{m x_0^2}$$

The total energy is constant, given by (6), and so,

$$\begin{aligned} E_n &= \frac{1}{2} m x_0^2 \omega^2 \\ &= \frac{1}{2} m x_0^2 \omega \frac{2n\hbar}{m x_0^2} \\ &= n\hbar\omega \end{aligned}$$

This is very close to the LHO as developed by solving the Schroedinger equation using the above potential, viz.

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$