Mechanics with fractional derivatives

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Lagrangian and Hamiltonian mechanics can be formulated to include derivatives of fractional order [F. Riewe, Phys. Rev. **53**, 1890 (1996)]. Lagrangians with fractional derivatives lead directly to equations of motion with nonconservative classical forces such as friction. The present work continues the development of fractional-derivative mechanics by deriving a modified Hamilton's principle, introducing two types of canonical transformations, and deriving the Hamilton-Jacobi equation using generalized mechanics with fractional and higher-order derivatives. The method is illustrated with a frictional force proportional to velocity. In contrast to conventional mechanics with integer-order derivatives, quantization of a fractional-derivative Hamiltonian cannot generally be achieved by the traditional replacement of momenta with coordinate derivatives. Instead, a quantum-mechanical wave equation is proposed that follows from the Hamilton-Jacobi equation by application of the correspondence principle. [S1063-651X(97)01403-7]

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I. INTRODUCTION

In 1931, Bauer [1] proved that it is impossible to use a variational principle to derive a single linear dissipative equation of motion with constant coefficients. Bauer's theorem expresses the well-known belief that there is no direct method of applying variational principles to nonconservative systems, which are characterized by friction or other dissipative processes. As stated by Lanczos [2], ''Forces of a frictional nature ... are outside the realm of variational principles, while the Newtonian scheme has no difficulty in including them.'' The techniques of Lagrangian and Hamiltonian mechanics, which are derived from variational principles, thus appears to be out of reach.

The proof of Bauer's theorem, however, relies on the tacit assumption that all derivatives are of integer order. If a Lagrangian is constructed using noninteger-order derivatives, then the resulting equation of motion can be nonconservative [3]. Because most classical processes observed in the physical world are nonconservative, it is important to be able to apply the power of variational methods to such cases. Nonconservative quantum processes are common too, since there is dissipation in every nonequilibrium or fluctuating process, including tunneling [4], electromagnetic cavity radiation [5,6], masers and parametric amplification [6], Brownian motion [7,8], inelastic scattering [9,10], squeezed states of quantum optics [11], and electrical resistance or Ohmic friction [12]. Since the starting point for the quantum-mechanical treatment of a phenomenon is usually the Hamiltonian or a related function, variational principles are important here too.

Besides the use of fractional derivatives, a number of other methods have been proposed that take advantage of loopholes in Bauer's [1] theorem. Bateman [13] suggested two methods based on the idea that a Lagrangian could lead to multiple equations. His first technique introduces an auxiliary coordinate y that describes a reverse-time system with negative friction. The Lagrangian for the combined system is

$$L=m\dot{x}\dot{y}+\tfrac{1}{2}\gamma(x\dot{y}-\dot{x}y), \tag{1}$$

which leads to two equations of motion

$$m\ddot{x}+\gamma\dot{x}=0, \quad m\ddot{y}-\gamma\dot{y}=0. \tag{2}$$

Here, dots indicate time derivatives. Even though the first equation describes a frictional force, the corresponding Hamiltonian leads to extraneous solutions that must be suppressed and the physical meaning of the momenta is unclear. The method is also described by Morse and Feshbach [14] and has been used in several applications [11,15].

Dekker [16] has added a clever twist to the auxiliary-coordinate method. He considers a Lagrangian which provides two first-order equations that are complex conjugates of each other, so that there is no nonphysical auxiliary equation. The equations can be combined to form a real, second-order equation of motion. Dekker's report [16] also provides a comprehensive review of work related to dissipation in Lagrangian and Hamiltonian mechanics.

Bateman's second method uses a Lagrangian that leads to an Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion. For example, the time-dependent Lagrangian

$$L=\tfrac{1}{2}m\dot{x}^2e^{(\gamma/m)t} \tag{3}$$

leads to the Euler-Lagrange equation

$$e^{(\gamma/m)t}(m\ddot{x}+\gamma\dot{x})=0. \tag{4}$$

The desired equation of motion is obtained if the factor $e^{(\gamma/m)t}$ is ignored. However, the corresponding momentum and Hamiltonian do not appear to be physically meaningful. Also, Ray [17] has shown that the Lagrangian should be interpreted as describing a system with increasing mass, rather than one with dissipation. Other work using this method can be found in Refs. [6,8,18].

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An approach that applies only to quantum systems is to modify the Schrödinger equation directly [8,10,19]. For example, a nonlinear term proportional to $\ln(\psi/\psi^*)$ can be added, sometimes accompanied by a second term to ensure conservation of energy. These types of modifications provide quantum results corresponding to classical friction, although many appear to have other unsatisfactory or peculiar features [10]. This method does not correspond to any classical technique for including friction in Lagrangian and Hamiltonian mechanics.

A standard device for dealing with dissipation is the Rayleigh dissipation function (Ref. [20], pp. 21 and 22), which can be used when frictional forces are proportional to velocity. For a particle in one dimension, Rayleigh's function is

$$\mathcal{F}=\tfrac{1}{2}\gamma\dot{x}^2 \tag{5}$$

and Lagrange's equation must be rewritten in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}+\frac{\partial \mathcal{F}}{\partial \dot{x}}=0. \tag{6}$$

In this case, it takes two scalar functions to specify the equation of motion. The momentum and the Hamiltonian are the same as if no friction were present, so they are of no use when attempting to add friction to Hamiltonian mechanics or quantum theory.

The most realistic approach is to include the microscopic details of the dissipation directly in the Lagrangian or Hamiltonian [4,5,7,9,12,21]. For example, if the dissipation is due to the interaction with a bath of harmonic oscillators with coordinates y_j, the following Lagrangian can be used:

$$L=\frac{1}{2}m\dot{x}^2-V(x)+\sum_j \frac{1}{2}m_j\left[\dot{y}_j^2-\omega_j^2\left(y_j-\frac{c_j}{m_j\omega_j^2}x\right)^2\right]. \tag{7}$$

This method is well suited to a wide range of realistic applications that can be modeled with harmonic oscillators. However, it is not intended to be a general method for introducing friction into classical Lagrangian mechanics. It can be complex in practice and does not allow the functional form of the frictional force to be chosen arbitrarily.

The techniques described above are not as simple and direct as conservative mechanics. To put the mechanics of nonconservative systems on the same footing as the conservative, a method was presented in Ref. [3], and is extended in the present paper, that allows nonconservative forces to be calculated directly from a Lagrangian. Hamilton's equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. The method is suggested by the observation that a term proportional to the nth-order derivative d^nx/dt^n in the Euler-Lagrange equation follows from a Lagrangian with a term proportional to $(d^{n/2}x/dt^{n/2})^2$. Hence a frictional force of the form $\gamma(dx/dt)$ might follow directly from a Lagrangian containing a term with the half-order derivative $(d^{1/2}x/dt^{1/2})^2$. It was shown in Ref. [3] that such fractional derivatives in the Lagrangian do indeed describe nonconservative forces. This technique overcomes many of the objections raised for the other methods, but its price is the complexity and unfamiliarity of fractional calculus.

Derivatives of any noninteger order are usually termed ''fractional derivatives.'' Since the mathematical techniques for dealing with derivatives of noninteger order are relatively unfamiliar, fractional calculus is reviewed in Sec. II. The methods of Lagrangian mechanics with fractional derivatives from Ref. [3] are reviewed in Sec. III. Section IV provides a derivation of Hamilton's equations using a modified Hamilton's principle. Mechanics with fractional derivatives is extended to included canonical transformations in Sec. V and Hamilton-Jacobi theory in Sec. VI. In Sec. VII the methods are illustrated using the example of a classical frictional force proportional to velocity. An appropriate quantization procedure is then presented in Sec. VIII. In Sec. IX fractional mechanics is used to solve a puzzle first published in 1931. Conclusions are presented in Sec. X.

II. FRACTIONAL CALCULUS

A review of fractional calculus was presented in Ref. [3], as was a brief history of the subject. Additional details can be found in textbooks [22,23] and a recent review article [24]. This section provides a summary of the needed mathematics.

The fractional integral of order ν is defined by

$$\frac{d^{-\nu}f(t)}{d(t-c)^{-\nu}}=\frac{1}{\Gamma(\nu)}\int_c^t (t-t')^{\nu-1}f(t')dt' \qquad [\,\mathrm{Re}(\nu)>0]. \tag{8}$$

If n is the smallest integer greater than Re (u), and $\nu=n-u$, then the fractional derivative of order u is defined by

$$\frac{d^uf(t)}{d(t-c)^u}=\frac{d^n}{dt^n}\frac{d^{-\nu}f(t)}{d(t-c)^{-\nu}}. \tag{9}$$

For integer values of u, the definition reduces to the usual definition of derivative.

The above notation, which will be used throughout the paper, follows Oldham and Spanier [22]. Another common notation was introduced by Davis [25] and is used by Miller and Ross [23]:

$${}_cD_t^uf(t)=\frac{d^uf(t)}{d(t-c)^u}. \tag{10}$$

A definition, especially useful when $t<c$, is the Weyl derivative

$${}_cW_t^uf(t)=(-1)^{-u}\frac{d^uf(t)}{d(t-c)^u}. \tag{11}$$

Use of the Weyl derivative would simplify certain formulas in this paper by eliminating the sometimes ambiguous factor $(-1)^{-u}$. However, the notation is less physically meaningful, since the Weyl derivative reduces to the negative of the ordinary derivative when u is an odd integer. All of the above notations emphasize that the fractional derivative of a function is not determined by the behavior of the function at the single value t, but depends on the values of the function over the entire interval c to t, just as a definite integral depends on values throughout the interval of integration.

It is illuminating to consider the special case $c=0$. Then the expression for the derivative of a power of t has the same form as for integer-order derivatives:

$$\frac{d^u t^\nu}{dt^u}=\frac{\nu!}{(\nu-u)!}t^{\nu-u}. \tag{12}$$

The factorials $\nu!$ and $(\nu-u)!$ must now be interpreted as the gamma functions $\Gamma(\nu+1)$ and $\Gamma(\nu-u+1)$. Another special case is $c=-\infty$, for which

$$\frac{d^u e^{at}}{d(t+\infty)^u}=a^u e^{at}, \tag{13}$$

which is the same as the expression for integer-order derivatives. Equations (12) and (13) can be used to calculate the fractional derivatives of any functions that can be expressed as sums of powers or exponentials.

In this paper integer-order derivatives with respect to t may be denoted with dots, so that $\dot{x}=dx/dt$ and $\ddot{x}=d^2x/dt^2$. Derivatives of arbitrary order with respect to t are sometimes indicated by a subscript or superscript in parentheses: $x_{(u,a)}=x^{(u,a)}=d^u x/d(t-a)^u$. The constant may be omitted if it is zero or if its value is clear from the context.

A formula we will need is integration by parts of a fractional derivative. The conventional formula for integer-order derivatives is

$$\int_a^b \frac{d^n f(t)}{dt^n}g(t)dt-(-1)^{-n}\int_a^b f(t)\frac{d^n g(t)}{dt^n}\,dt$$

$$=\sum_{k=0}^{n-1}(-1)^k\frac{d^{n-k-1}f(t)}{dt^{n-k-1}}\left.\frac{d^k g(t)}{dt^k}\right|_a^b. \tag{14}$$

Love and Young [26] have obtained a fractional-order formula

$$\int_a^b \frac{d^{-\nu}f(t)}{d(t-a)^{-\nu}}g(t)dt=(-1)^\nu\int_a^b f(t)\frac{d^{-\nu}g(t)}{d(t-b)^{-\nu}}dt \tag{15}$$

for $0<\nu<1$. To obtain a general formula for integration by parts for order u, we choose n to be the smallest integer greater than u, and let $\nu=n-u$. Then application of Eq. (14) followed by Eq. (15) yields the general formula

$$\int_a^b \frac{d^u f(t)}{dt^u}g(t)dt-(-1)^{-u}\int_a^b f(t)\frac{d^u g(t)}{dt^u}\,dt$$

$$=\sum_{k=0}^{n-1}(-1)^k\frac{d^{n-k-1}f(t)}{dt^{n-k-1}}\left.\frac{d^k g(t)}{dt^k}\right|_a^b. \tag{16}$$

When $d^k f/dt^k=0$ or $d^k g/dt^k=0$ for $k=0$ to $n-1$, this becomes the result we will use,

$$\int_a^b \frac{d^u f(t)}{d(t-a)^u}g(t)dt=(-1)^{-u}\int_a^b f(t)\frac{d^u g(t)}{d(t-b)^u}dt. \tag{17}$$

III. GENERALIZED MECHANICS WITH FRACTIONAL DERIVATIVES

This section provides background and notation. It also reviews the results of Ref. [3], which introduced the generalization of mechanics to include derivatives of noninteger order. Section IV begins the presentation of material leading to Hamilton-Jacobi theory and a corresponding quantum-mechanical wave equation.

A. Background

In traditional Newtonian mechanics, equations of motion normally have derivatives of first or second order only. The corresponding Lagrangians have derivatives of only first order. Ostrogradsky [27] published a generalization of Lagrangian and Hamiltonian mechanics to include derivatives of arbitrarily high (integer) order. Such dynamical equations with higher-order derivatives can be used to describe particles with internal structure, such as spin or internal motion [28]. The formalism was extended to quantum electrodynamics by Bopp [29] and Podolsky [30] and to quantum field theory by Green [31]. Generalized mechanics is reviewed in Ref. [32] and recent applications are described in Refs. [33,34]. The present work and Ref. [3] can be considered to be a further generalization of mechanics to include noninteger derivatives of all orders.

B. Notation

The Lagrangian for generalized mechanics is a function of coordinates x_r, the time parameter t, and derivatives of x_r with respect to t. The subscript $r=1,\ldots,R$ indicates the particular coordinate (for example, $x_1=x$, $x_2=y$, $x_3=z$). The order of derivatives can be any non-negative real order, although in principle there is no reason to exclude more general derivatives, such as complex order. If the Lagrangian is a function of the coordinate x_r and N different derivatives of x_r, then we will use $s(n)$ to indicate the order of the nth derivative, where $n=1,\ldots,N$. For example, if the lowest-order derivative is $d^{1/2}x/d(t-b)^{1/2}$, then $s(1)=\frac{1}{2}$. In Ref. [3] it was found that if the fractional calculus of variations is applied over the time interval $t=a$ to b, then the Lagrangian can contain two types of derivatives:

$$q_{r,s(n)}=q_{r,s(n),b}=\frac{d^{s(n)}x_r}{d(t-b)^{s(n)}} \tag{18}$$

and

$$q_{r,s'(n),a}=\frac{d^{s'(n)}x_r}{d(t-a)^{s'(n)}}, \tag{19}$$

where $s(n)$ and $s'(n)$ can be any non-negative real numbers (or complex numbers with $\mathrm{Re}[s(n)]\geqslant 0$). We define $s(0)$ to be 0, so that $q_{r,s(0)}$ denotes the coordinate x_r. The subscript r may sometimes be omitted. For some applications, it may be more convenient to streamline the notation by writing $q_{r,n}$ or to number all coordinates consecutively: q_i.

As in Ref. [3], we will simplify the derivations by using a Lagrangian that does not contain any derivatives with respect to $t-a$. The straightforward extension of each final result to

Lagrangians with both types of derivatives will then be provided. In derivations that use only coordinates defined by Eq. (18), the subscript b on the coordinates will be omitted.

The notation $L(\{q_{r,s(n)}\},t)$ will be used to indicate that the Lagrangian is a function of the parameter t and the set of all $q_{r,s(n)}$ for $r=1,\ldots,R$ and $n=0,\ldots,N$. The notation $L(\{q_{r,s'(n),a},q_{r,s(n),b}\},t)$ designates a Lagrangian that is a function of both types of coordinates. Because summations over r will always be over all values, we will use the usual convention of summing over repeated indices. However, we will not be able to use the summation convention for n in all cases, so all summations over n will be indicated explicitly.

C. Euler-Lagrange equation

The Euler-Lagrange equation was derived two different ways in Ref. [3]. The first was a generalization of Euler's original (integer-order) method based on finite differences and the second followed the same pattern as in conventional classical mechanics (Ref. [20], Chap. 2) by developing and applying fractional calculus of variations. The use of calculus of variations avoids ambiguities of Euler's method, such as exchanging the order of limits and summation. The end points of the integration interval are chosen to be fixed, so that we can exchange the order of integration and differentiation. The path is varied, but not the time, so we can exchange the order of differentiation.

By varying the integral

$$J=\int_a^b L(\{q_{r,s'(n),a},q_{r,s(n),b}\},t)dt, \tag{20}$$

and using fractional integration by parts, Eq. (17), it was shown in Ref. [3] that we obtain the Euler-Lagrange equation

$$\sum_{n=0}^{N}(-1)^{s(n)}\frac{d^{s(n)}}{d(x-a)^{s(n)}}\frac{\partial L}{\partial q_{r,s(n),b}} + \sum_{n=1}^{N'}(-1)^{-s'(n)}\frac{d^{s'(n)}}{d(x-b)^{s'(n)}}\frac{\partial L}{\partial q_{r,s'(n),a}}=0. \tag{21}$$

D. Hamilton's equations

In Ref. [3], Hamilton's equations were derived using the Euler-lagrange equation. An alternative method of obtaining the same result uses a variational principle. The derivation will be presented in Sec. IV.

IV. MODIFIED HAMILTON'S PRINCIPLE

The modified Hamilton's principle is a variational principle that yields Hamilton's equations. It is the launching point for canonical transformations and Hamilton-Jacobi theory. The derivation here with fractional derivatives follows the conventional method given by Goldstein (Ref. [20], p. 225). Using the procedure described in Sec. III B, we will initially assume that L contains only coordinates $q_{r,s(n),b}$ as defined in Eq. (18) and the results will then be extended to also include coordinates $q_{r,s'(n),a}$ from Eq. (19).

As in Ref. [3], define the momenta

$$\begin{aligned} p_{r,s(n)} &= p_{r,s(n),b} \\ &= \sum_{k=0}^{N-n-1}(-1)^{s(k+n+1)-s(n+1)} \\ &\quad \times \frac{d^{s(k+n+1)-s(n+1)}}{d(t-a)^{s(k+n+1)-s(n+1)}} \\ &\quad \times \frac{\partial L}{\partial q_{r,s(k+n+1)}}, \end{aligned} \tag{22}$$

where $n=0,\ldots,N-1$. It is also possible to define the momenta recursively by

$$p_{r,s(N-1)}=\frac{\partial L}{\partial q_{r,s(N)}} \tag{23}$$

and

$$p_{r,s(k-1)}=(-1)^{s(k+1)-s(k)}\frac{d^{s(k+1)-s(k)}}{d(t-a)^{s(k+1)-s(k)}}p_{r,s(k)} + \frac{\partial L}{\partial q_{r,s(k)}} \quad (k=1,\ldots,N-1). \tag{24}$$

The Hamiltonian is

$$H=\sum_{n=1}^{N} q_{r,s(n)}p_{r,s(n-1)}-L, \tag{25}$$

where the summation convention implies summation over r.

The variational principle

$$\delta J(\alpha)=\delta\int_a^b L(\{q_{r,s(n)}(t,\alpha)\},t)dt=0 \tag{26}$$

can be rewritten in terms of the Hamiltonian as

$$\delta I(\alpha)=\delta\int_a^b\left(\sum_{n=1}^{N} q_{r,s(n)}p_{r,s(n-1)}-H\right)dt=0. \tag{27}$$

Then

$$\begin{aligned} 0=\delta I(\alpha) &= \frac{\partial I}{\partial\alpha}d\alpha \\ &= d\alpha\int_a^b\sum_{n=0}^{N-1}\left(\frac{\partial q_{r,s(n+1)}}{\partial\alpha}p_{r,s(n)}+q_{r,s(n+1)}\frac{\partial p_{r,s(n)}}{\partial\alpha}\right. \\ &\quad \left. -\frac{\partial H}{\partial q_{r,s(n)}}\frac{\partial q_{r,s(n)}}{\partial\alpha}-\frac{\partial H}{\partial p_{r,s(n)}}\frac{\partial p_{r,s(n)}}{\partial\alpha}\right)dt. \end{aligned} \tag{28}$$

In terms of $q_{r,s(n)}$, rather than $q_{r,s(n+1)}$, this becomes

$$0=\delta I(\alpha)=\frac{\partial I}{\partial\alpha}d\alpha$$
$$=d\alpha\int_a^b\sum_{n=0}^{N-1}\left(p_{r,s(n)}\frac{d^{s(n+1)-s(n)}}{d(t-b)^{s(n+1)-s(n)}}\frac{\partial q_{r,s(n)}}{\partial\alpha}+\frac{d^{s(n+1)-s(n)}q_{r,s(n)}}{d(t-b)^{s(n+1)-s(n)}}\frac{\partial p_{r,s(n)}}{\partial\alpha}-\frac{\partial H}{\partial q_{r,s(n)}}\frac{\partial q_{r,s(n)}}{\partial\alpha}-\frac{\partial H}{\partial p_{r,s(n)}}\frac{\partial p_{r,s(n)}}{\partial\alpha}\right)dt$$
$$=d\alpha\int_a^b\sum_{n=0}^{N-1}\left((-1)^{s(n+1)-s(n)}\frac{d^{s(n+1)-s(n)}p_{r,s(n)}}{d(t-a)^{s(n+1)-s(n)}}\frac{\partial q_{r,s(n)}}{\partial\alpha}+\frac{d^{s(n+1)-s(n)}q_{r,s(n)}}{d(t-b)^{s(n+1)-s(n)}}\frac{\partial p_{r,s(n)}}{\partial\alpha}\right.$$
$$\left.-\frac{\partial H}{\partial q_{r,s(n)}}\frac{\partial q_{r,s(n)}}{\partial\alpha}-\frac{\partial H}{\partial p_{r,s(n)}}\frac{\partial p_{r,s(n)}}{\partial\alpha}\right)dt, \tag{29}$$

where the last step used integration by parts, Eq. (17). Next define the variations

$$\delta q_{r,s(n)}=\frac{\partial q_{r,s(n)}}{\partial\alpha}d\alpha \tag{30}$$

and

$$\delta p_{r,s(n)}=\frac{\partial p_{r,s(n)}}{\partial\alpha}d\alpha. \tag{31}$$

Then we have

$$0=\int_a^b\sum_{n=0}^{N-1}\left((-1)^{s(n+1)-s(n)}\frac{d^{s(n+1)-s(n)}p_{r,s(n)}}{d(t-a)^{s(n+1)-s(n)}}\delta q_{r,s(n)}+\frac{d^{s(n+1)-s(n)}q_{r,s(n)}}{d(t-b)^{s(n+1)-s(n)}}\delta p_{r,s(n)}-\frac{\partial H}{\partial q_{r,s(n)}}\delta q_{r,s(n)}-\frac{\partial H}{\partial p_{r,s(n)}}\delta p_{r,s(n)}\right)dt$$
$$=\int_a^b\sum_{n=0}^{N-1}\left[\left((-1)^{s(n+1)-s(n)}\frac{d^{s(n+1)-s(n)}p_{r,s(n)}}{d(t-a)^{s(n+1)-s(n)}}-\frac{\partial H}{\partial q_{r,s(n)}}\right)\delta q_{r,s(n)}+\left(\frac{d^{s(n+1)-s(n)}q_{r,s(n)}}{d(t-b)^{s(n+1)-s(n)}}-\frac{\partial H}{\partial p_{r,s(n)}}\right)\delta p_{r,s(n)}\right]dt. \tag{32}$$

Since the variations $\delta q_{r,s(n)}$ and $\delta p_{r,s(n)}$ are independent, we obtain Hamilton's canonical equations:

$$\frac{\partial H}{\partial q_{r,s(n)}}=(-1)^{s(n+1)-s(n)}\frac{d^{s(n+1)-s(n)}}{d(t-a)^{s(n+1)-s(n)}}p_{r,s(n)},$$
$$\frac{\partial H}{\partial p_{r,s(n)}}=q_{r,s(n+1)}, \tag{33}$$
$$\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}.$$

If the Lagrangian is a function of coordinates defined by both Eqs. (18) and (19), then we must define additional momenta

$$p_{r,s'(n),a}=\sum_{k=0}^{N'-n-1}(-1)^{-[s'(k+n+1)-s'(n+1)]}\times\frac{d^{s'(k+n+1)-s'(n+1)}}{d(t-b)^{s'(k+n+1)-s'(n+1)}}\left(\frac{\partial L}{\partial q_{r,s'(k+n+1),a}}\right), \tag{34}$$

where $n=0,\ldots,N'-1$. The Hamiltonian is then

$$H=\sum_{n=1}^{N}q_{r,s(n),b}p_{r,s(n-1),b}+\sum_{n=1}^{N'}q_{r,s'(n),a}p_{r,s'(n-1),a}-L. \tag{35}$$

For this Hamiltonian, we have the additional equations

$$\frac{\partial H}{\partial q_{r,s'(n),a}}=(-1)^{-[s'(n+1)-s'(n)]}\times\frac{d^{s'(n+1)-s'(n)}}{d(t-b)^{s'(n+1)-s'(n)}}p_{r,s'(n),a}, \tag{36}$$
$$\frac{\partial H}{\partial p_{r,s'(n),a}}=q_{r,s'(n+1),a}.$$

These results are the same as were derived using the Euler-Lagrange equation in Ref. [3]. The present method of derivation is needed as a basis for the canonical transformations in Sec. V.

For integer-order derivatives, it can be shown (Ref. [20], p. 220) that

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}. \tag{37}$$

Hence if the Lagrangian is not an explicit function of time and all derivatives are of integer order, then the Hamiltonian is a constant of the motion. However, if there are noninteger-order derivatives in the Lagrangian, it was shown in Ref. [3] that Eq. (37) does not hold and therefore a Hamiltonian with fractional derivatives is not generally a constant of the motion and the system is nonconservative.

V. CANONICAL TRANSFORMATIONS

Transformations are called canonical if they preserve the form of Hamilton's canonical equations. They transform the coordinates $q_{s(i)}$ and momenta $p_{s(i)}$ into new variables $Q_{S(i)}(q,p,t)$ and $P_{S(i)}(q,p,t)$ and provide a new function $K(Q,P,t)$ that plays the part of the Hamiltonian. For simplicity, subscripts will usually be omitted when showing functional dependence of K and the subscript r will be omitted for coordinates and momenta. The derivations will be extensions of the method used by Goldstein (Ref. [20], Chap. 8) for conventional mechanics.

To be canonical, new coordinates must satisfy a modified Hamilton's principle of the form

$$\delta\int_a^b\left(\sum_{i=0}^{N-1} P_{S(i)}Q_{S(i+1)}-K(Q,P,t)\right)dt=0. \qquad (38)$$

At the same time, the original coordinates satisfy the similar principle

$$\delta\int_a^b\left(\sum_{i=0}^{N-1} p_{s(i)}q_{s(i+1)}-H(q,p,t)\right)dt=0. \qquad (39)$$

For these equations to hold, the integrands must differ by the total time derivative of an arbitrary function F:

$$\left(\sum_{i=0}^{N-1} P_{S(i)}Q_{S(i+1)}-K(Q,P,t)\right) - \left(\sum_{i=0}^{N-1} p_{s(i)}q_{s(i+1)}-H(q,p,t)\right)=\frac{dF}{dt}. \qquad (40)$$

This relation follows from

$$\int_a^b\frac{dF}{dt}dt=F(b)-F(a). \qquad (41)$$

Since F is not varied at the end points, we automatically get

$$\delta\int_a^b\frac{dF}{dt}dt=\delta[F(b)-F(a)]=0. \qquad (42)$$

The function F, which completely determines the transformation, is called the generating function. For F to produce a transformation from one variable to another, it must be a function of both the old and new variables. We thus have four traditional forms for F: $F_1(q,Q,t)$, $F_2(q,P,t)$, $F_3(p,Q,t)$, and $F_4(p,P,t)$.

For mechanics with fractional derivatives, these generating functions will be seen to lead to awkward expressions involving fractional time derivatives. However, satisfactory transformations can be obtained using generating functions with different variables $\overline{q_{s(i)}}$ and $\overline{Q_{S(i)}}$, defined by

$$\frac{d\overline{q_{s(i)}}}{dt}=q_{s(i+1)}, \qquad \frac{d\overline{Q_{S(i)}}}{dt}=Q_{S(i+1)}. \qquad (43)$$

For integer-order derivatives, these new coordinates are the same as the usual canonical coordinates. However, when dealing with fractional derivatives, the coordinates $\overline{q_{s(i)}}$ and $\overline{Q_{S(i)}}$ will not be canonical, so all canonical expressions must be written in terms of the original coordinates $q_{s(i)}$ and $Q_{S(i)}$. The four kinds of generating functions are then $\overline{F_1}(\overline{q},\overline{Q},t)$, $\overline{F_2}(\overline{q},P,t)$, $F_3(p,\overline{Q},t)$, and $F_4(p,P,t)$. For our present purposes, there is no need to deal with F_3 or F_4. We will refer to transformations using the original canonical coordinates as ''direct'' transformations, to distinguish them from the transformations using the $\overline{q_{s(i)}}$ and $\overline{Q_{S(i)}}$. We will first derive the canonical transformations for $\overline{F_1}$ and $\overline{F_2}$ and then state the results for the less useful direct canonical transformations.

A. Canonical transformations of the first kind

For a generating function $\overline{F_1}(\overline{q},\overline{Q},t)$ that is a function of $\overline{q_{s(i)}}$ and $\overline{Q_{S(i)}}$, the transformation is

$$\begin{aligned}
&\left(\sum_{i=0}^{N-1} p_{s(i)}q_{s(i+1)}-H\right)-\left(\sum_{i=0}^{N-1} P_{S(i)}Q_{S(i+1)}-K\right)\\
&\quad=\frac{d}{dt}\overline{F_1}(\overline{q},\overline{Q},t)\\
&\quad=\sum_{i=0}^{N-1}\frac{\partial\overline{F_1}}{\partial\overline{q_{s(i)}}}\frac{d\overline{q_{s(i)}}}{dt}+\sum_{i=0}^{N-1}\frac{\partial\overline{F_1}}{\partial\overline{Q_{S(i)}}}\frac{d\overline{Q_{S(i)}}}{dt}+\frac{\partial\overline{F_1}}{\partial t}\\
&\quad=\sum_{i=0}^{N-1}\frac{\partial\overline{F_1}}{\partial\overline{q_{s(i)}}}q_{s(i+1)}+\sum_{i=0}^{N-1}\frac{\partial\overline{F_1}}{\partial\overline{Q_{S(i)}}}Q_{S(i+1)}+\frac{\partial\overline{F_1}}{\partial t}.
\end{aligned} \qquad (44)$$

Since the variables q and Q are considered to be independent, the equation can hold identically only if the coefficients of $q_{s(i+1)}$ and $Q_{S(i+1)}$ are each equal to zero. Hence we have the transformation equations

$$p_{s(i)}=\frac{\partial\overline{F_1}}{\partial\overline{q_{s(i)}}},\qquad P_{S(i)}=\frac{\partial\overline{F_1}}{\partial Q_{S(i)}},\qquad K=H+\frac{\partial\overline{F_1}}{\partial t}. \qquad (45)$$

B. Canonical transformations of the second kind

Following the traditional method for integer-order derivatives, we can define the generating function $\overline{F_2}(\overline{q},P,t)$ as the Legendre transformation

$$\overline{F_2}(\overline{q},P,t)=\overline{F_1}(\overline{q},\overline{Q},t)+\sum_{i=0}^{N-1}P_{S(i)}\overline{Q_{S(i)}}. \tag{46}$$

Solve this for $\overline{F_1}$ and substitute into Eq. (44) to get

$$\begin{aligned}\left(\sum_{i=0}^{N-1}p_{s(i)}q_{s(i+1)}-H\right)&=\left(\sum_{i=0}^{N-1}P_{S(i)}Q_{S(i+1)}-K\right)\\&\quad+\frac{d}{dt}\left(\overline{F_2}-\sum_{i=0}^{N-1}P_{S(i)}\overline{Q_{S(i)}}\right)\\&=-K+\frac{d}{dt}\overline{F_2}-\sum_{i=0}^{N-1}\frac{dP_{S(i)}}{dt}\overline{Q_{S(i)}}\\&=-K+\sum_{i=0}^{N-1}\frac{\partial\overline{F_2}}{\partial\overline{q_{s(i)}}}q_{s(i+1)}\\&\quad+\sum_{i=0}^{N-1}\frac{\partial\overline{F_2}}{\partial P_{S(i)}}\frac{dP_{S(i)}}{dt}+\frac{\partial\overline{F_2}}{\partial t}\\&\quad-\sum_{i=0}^{N-1}\frac{dP_{S(i)}}{dt}\overline{Q_{S(i)}}.\end{aligned} \tag{47}$$

Equating the coefficients of $q_{s(i+1)}$ and $dP_{S(i)}/dt$ gives us the transformation equations

$$\begin{aligned}p_{s(i)}&=\frac{\partial\overline{F_2}}{\partial\overline{q_{s(i)}}},\\\overline{Q_{S(i)}}&=\frac{\partial\overline{F_2}}{\partial P_{S(i)}},\\K&=H+\frac{\partial\overline{F_2}}{\partial t}.\end{aligned} \tag{48}$$

C. Direct canonical transformations

The most direct derivation of the canonical transformations does not use the auxiliary coordinates $\overline{q_{s(i)}}$ and $\overline{Q_{S(i)}}$. However, if only the canonical coordinates are used, certain terms can only be made to cancel by keeping all terms under the variational integral, as with Eqs. (38) and (39), and using integration by parts. As seen from Eq. (17), integration by parts introduces fractional time derivatives that complicate the transformation equations. These time derivatives cause difficulties with the Hamilton-Jacobi and wave equations. Since derivations of the direct transformations follow the same pattern as used in Secs. V A and V B, the derivations will not be shown. The direct canonical transformation of the first kind is

$$\begin{aligned}(-1)^{[s(i+1)-1]}\frac{d^{[s(i+1)-1]}p_{s(i)}}{d^{[s(i+1)-1]}(t-a)}&=\frac{\partial F_1}{\partial q_{s(i)}},\\(-1)^{[S(i+1)-1]}\frac{d^{[S(i+1)-1]}P_{S(i)}}{d^{[S(i+1)-1]}(t-a)}&=\frac{\partial F_1}{\partial Q_{S(i)}},\\K&=H+\frac{\partial F_1}{\partial t}\end{aligned} \tag{49}$$

and the direct canonical transformation of the second kind is

$$\begin{aligned}\frac{d^{[S(i+1)-S(i)]-1}}{d(t-b)^{[S(i+1)-S(i)]-1}}Q_{S(i)}&=\frac{\partial F_2}{\partial P_{S(i)}},\\p_{s(i)}&=(-1)^{1-[s(i+1)-s(i)]}\frac{d^{1-[s(i+1)-s(i)]}}{d(t-a)^{1-[s(i+1)-s(i)]}}\frac{\partial F_2}{\partial q_{s(i)}},\\K&=H+\frac{\partial F_2}{\partial t}.\end{aligned} \tag{50}$$

In this paper these direct transformations will only be used for purposes of comparison.

VI. HAMILTON-JACOBI THEORY

As in conventional mechanics, the Hamilton-Jacobi equation results from a canonical transformation for which the new variables are constant in time. For integer-order derivatives, such a transformation will follow automatically if the new Hamiltonian K is identically zero, since from the equations of motion we then have

$$\begin{aligned}\dot{Q}_i&=\frac{\partial K}{\partial P_i}=0,\\\dot{P}_i&=-\frac{\partial K}{\partial Q_i}=0.\end{aligned} \tag{51}$$

For fractional derivatives satisfying $S(i+1)-S(i)\leqslant 1$, we can derive a similar relationship from

$$\begin{aligned}\frac{d^{[S(i+1)-S(i)]}}{d(t-b)^{[S(i+1)-S(i)]}}Q_{S(i)}&=\frac{\partial K}{\partial P_{S(i)}}=0,\\(-1)^{[S(i+1)-S(i)]}\frac{d^{[S(i+1)-S(i)]}}{d(t-a)^{[S(i+1)-S(i)]}}P_{S(i)}&=\frac{\partial K}{\partial Q_{S(i)}}=0\end{aligned} \tag{52}$$

if we differentiate each side by $(-1)^{1-[S(i+1)-(S(i)]}[d^{1-[S(i+1)-S(i)]}/d(t-a)^{1-[S(i+1)-S(i)]}]$ or $d^{1-[S(i+1)-S(i)]}/d(t-b)^{1-[S(i+1)-S(i)]}$ to get

$$\begin{aligned}\dot{Q}_{S(i)}&=\frac{d^{1-[S(i+1)-S(i)]}}{d(t-a)^{1-[S(i+1)-S(i)]}}0=0,\\\dot{P}_{S(i)}&=(-1)^{1-[S(i+1)-S(i)]}\frac{d^{1-[S(i+1)-S(i)]}}{d(t-a)^{1-[S(i+1)-S(i)]}}0=0.\end{aligned} \tag{53}$$

If necessary, intermediate coordinates may have to be defined to ensure that $S(i+1)-S(i)\leqslant 1$.

Since K is related to H by

$$K(Q,P,t)=H(q,p,t)+\frac{\partial\overline{F_2}}{\partial t}, \tag{54}$$

it follows that K will be zero if $\overline{F_2}$ satisfies

$$H(q,p,t)+\frac{\partial\overline{F_2}}{\partial t}=0. \tag{55}$$

It will be convenient to express p in terms of the transformation equation

$$p_{s(i)}=\frac{\partial\overline{F_2}}{\partial\overline{q_{s(i)}}}, \tag{56}$$

in which case we obtain the Hamilton-Jacobi equation

$$H\left(q_{s(i)},\frac{\partial\overline{F_2}}{\partial\overline{q_{s(i)}}},t\right)+\frac{\partial\overline{F_2}}{\partial t}=0. \tag{57}$$

The solution ($\overline{F_2}$ in this case) of the Hamilton-Jacobi equation is usually denoted by S and called Hamilton's principal function.

Note that $S(\overline{q},P,t)$ is a function of $\overline{q}$ rather than the canonical coordinate q. If we wish to find a direct Hamilton-Jacobi equation in terms of q, a similar derivation using a generating function $F_2(q,P,t)$ from Sec. V C yields the more complicated equation

$$H\left(q_{s(i)},(-1)^{1-[S(i+1)-S(i)]}\frac{d^{1-[S(i+1)-S(i)]}}{d(t-a)^{1-[S(i+1)-S(i)]}}\times\frac{\partial F_2(q,P,t)}{\partial q_{s(i)}},t\right)+\frac{\partial F_2(q,P,t)}{\partial t}=0. \tag{58}$$

We will not make use of this form of the equation, except for brief references in Secs. VII and VIII.

We know that since $\dot{P}_{S(i)}=0$, the momenta must be constant. Hence the solution of the Hamilton-Jacobi equation can be written as $S(\overline{q_{s(i)}},\alpha_{S(i)},t)$, where each $\alpha_{S(i)}$ is a constant. We then have

$$p_{s(i)}=\frac{\partial S(\overline{q_{s(i)}},\alpha_{S(i)},t)}{\partial\overline{q_{s(i)}}}. \tag{59}$$

The other transformation equation provides the new constant coordinates

$$\beta_{S(i)}=\overline{Q_{S(i)}}=\frac{\partial S(\overline{q_{s(i)}},\alpha_{S(i)},t)}{\partial\alpha_{S(i)}}. \tag{60}$$

This equation can be solved for $q(\alpha_{s(i)},\beta_{S(i)},t)$ to get the final solution to the problem.

An interesting result of conventional classical mechanics is that the solution S to the Hamilton-Jacobi equation equals, to within an additive constant, the integral $\int L dt$. We now show that the same holds true for the case of fractional derivatives. The time derivative of S can be written

$$\begin{aligned}\frac{dS}{dt}&=\sum_{i=0}^{N-1}\frac{\partial S}{\partial\overline{q_{s(i)}}}\frac{d\overline{q_{s(i)}}}{dt}+\sum_{i=0}^{N-1}\frac{\partial S}{\partial P_{S(i)}}\frac{dP_{S(i)}}{dt}+\frac{\partial S}{\partial t}\\&=\sum_{i=0}^{N-1}\frac{\partial S}{\partial\overline{q_{s(i)}}}\frac{d\overline{q_{s(i)}}}{dt}+\frac{\partial S}{\partial t}\\&=\sum_{i=0}^{N-1}\frac{\partial S}{\partial\overline{q_{s(i)}}}q_{s(i+1)}+\frac{\partial S}{\partial t},\end{aligned} \tag{61}$$

since the momenta are constant in time. By substituting from the Hamilton-Jacobi equation, we get

$$\frac{dS}{dt}=\sum_{i=0}^{N-1}\frac{\partial S}{\partial\overline{q_{s(i)}}}q_{s(i+1)}-H \tag{62}$$

and from

$$p_{s(i)}=\frac{\partial S(\overline{q_{s(i)}},\alpha_{S(i)},t)}{\partial\overline{q_{s(i)}}} \tag{63}$$

we find

$$\frac{dS}{dt}=\sum_{i=0}^{N-1}p_{s(i)}q_{s(i+1)}-H=L, \tag{64}$$

or

$$S=\int L dt+\text{ const.} \tag{65}$$

VII. APPLICATION TO LINEAR FRICTION

The formalism of the preceding sections can be illustrated with the example of a frictional force proportional to velocity. For simplicity, we will choose a Lagrangian that is a function of coordinates defined by Eq. (18). We will consider the limiting case in which $a\to b$ while keeping $a<b$, so that all fractional derivatives can be approximated by derivatives of the form $d^u/d(t-b)^u$.

The three terms in the Lagrangian

$$\begin{aligned}L&=\frac{1}{2}m\left(\frac{dx}{dt}\right)^2-V(x)+i\frac{1}{2}\gamma\left(\frac{d^{1/2}x}{d(t-b)^{1/2}}\right)^2\\&=\frac{1}{2}m\dot{x}^2-V(x)+i\frac{1}{2}\gamma x_{(1/2,b)}^2\end{aligned} \tag{66}$$

represent kinetic energy, potential energy, and linear friction energy. The methods of Secs. III–VI can be applied by choosing $N=2$, $s(0)=0$, $s(1)=\frac{1}{2}$, and $s(2)=1$. The Lagrangian can be written as a function of the generalized coordinates:

$$L=\tfrac{1}{2}mq_1^2-V(q_0)+i\tfrac{1}{2}\gamma q_{1/2}^2. \tag{67}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial q_0}+i\frac{d^{1/2}}{d(t-b)^{1/2}}\frac{\partial L}{\partial q_{1/2}}-\frac{d}{dt}\frac{\partial L}{\partial q_1}=0, \tag{68}$$

which becomes

$$m\ddot{x} = -\gamma\dot{x} - \frac{\partial V(x)}{\partial x}. \tag{69}$$

The momenta are

$$p_0 = \left(\frac{\partial L}{\partial q_{1/2}}\right) + i\frac{d^{1/2}}{d(t-b)^{1/2}}\left(\frac{\partial L}{\partial q_1}\right)$$

$$= i\gamma x_{(1/2,b)} + imx_{(3/2,b)}, \tag{70}$$

$$p_{1/2} = \left(\frac{\partial L}{\partial q_1}\right) = m\dot{x}.$$

The Hamiltonian is

$$H = q_{1/2}p_0 + q_1 p_{1/2} - L$$

$$= \frac{p_{1/2}^2}{2m} + q_{1/2}p_0 + V(q_0) - i\frac{1}{2}\gamma q_{1/2}^2 \tag{71}$$

and Hamilton's equations are

$$\frac{\partial H}{\partial q_0} = i\frac{d^{1/2}}{d(t-b)^{1/2}}p_0, \quad \frac{\partial H}{\partial p_0} = q_{1/2},$$
$$\frac{\partial H}{\partial q_{1/2}} = i\frac{d^{1/2}}{d(t-b)^{1/2}}p_{1/2}, \quad \frac{\partial H}{\partial p_{1/2}} = q_1. \tag{72}$$

The first of Hamilton's equations yields the Euler-Lagrange equation, the one to its right is an identity, and the remaining two equations are equivalent to the definition of the momenta.

If we define coordinates

$$\overline{q_0} = q_{(-1/2)} = \frac{1}{\Gamma(\frac{1}{2})}\int_b^t (t-t')^{-1/2}q(t')\,dt',$$
$$\overline{q_{1/2}} = q_0, \tag{73}$$

then the Hamilton-Jacobi equation can be obtained from Eq. (57). Written in terms of the canonical coordinates, the Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q_0}\right)^2 + q_{1/2}\frac{\partial S}{\partial q_{(-1/2)}} + V(q_0) - i\frac{1}{2}\gamma q_{1/2}^2 = -\frac{\partial S}{\partial t}. \tag{74}$$

The rule for finding the corresponding quantum-mechanical wave equation will be shown in Sec. VIII to be the same as in conventional mechanics. This rule yields

$$\left[-\hbar^2\frac{1}{2m}\frac{\partial^2}{\partial q_0^2} - i\hbar q_{1/2}\frac{\partial}{\partial q_{(-1/2)}} + V(q_0) - i\frac{1}{2}\gamma q_{1/2}^2\right]\psi$$

$$= i\hbar\frac{\partial\psi}{\partial t}. \tag{75}$$

The wave equation is just the conventional Schrödinger equation, but with two extra terms involving fractional derivatives.

The methods of Sec. VI also provide a direct Hamilton-Jacobi equation in terms of the canonical variables. From Eq. (58) we find

$$-\frac{1}{2m}\left(\frac{d^{1/2}}{d(t-a)^{1/2}}\frac{\partial S}{\partial q_{1/2}}\right)^2 + (-1)^{1/2}q_{1/2}\frac{d^{1/2}}{d(t-a)^{1/2}}\frac{\partial S}{\partial q_0}$$

$$+ V(q_0) - i\frac{1}{2}\gamma q_{1/2}^2 = -\frac{\partial S}{\partial t}. \tag{76}$$

As discussed earlier, this form of the equation is unsatisfactory for most purposes, due to the fractional time derivatives.

This simple example illustrates the basic techniques for applying fractional-derivative mechanics to linear friction, but it does not deal with more realistic scenarios that might include driving noise or more general frictional forces.

VIII. WAVE EQUATION

A. Wrong way

Our next goal is to find a quantum wave equation corresponding to a Hamiltonian with fractional derivatives. As a first guess, we might try the usual substitution,

$$p_{r,n} \to -i\hbar\frac{\partial}{\partial q_{r,n}} \tag{77}$$

to obtain the wave equation

$$H\left(q_{r,n}, -i\hbar\frac{\partial}{\partial q_{r,n}}\right)\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{78}$$

With a simple example, we can easily show that this rule may lead to the wrong result. Consider the Hamiltonian

$$H = \frac{P^2}{2m} + Qp + V(q) - i\frac{1}{2}\gamma Q^2. \tag{79}$$

Given this Hamiltonian, it is reasonable to think that it obeys Hamilton's equations

$$\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial P} = \dot{Q}, \quad \frac{\partial H}{\partial q} = -\dot{p}, \quad \frac{\partial H}{\partial Q} = -\dot{P}, \tag{80}$$

which lead to the equation of motion

$$\frac{\partial V}{\partial q} = m\ddddot{q} - i\gamma\ddot{q}. \tag{81}$$

If the Hamiltonian is quantized using Eq. (77), it is easy to see that the wave equation corresponds to this equation of motion.

Unfortunately, Eq. (79) is only a different notation for the fractional-derivative Hamiltonian given by Eq. (71), which corresponds to a different equation of motion,

$$\frac{\partial V}{\partial q} = -m\ddot{q} - \gamma\dot{q}. \tag{82}$$

In order to obtain Eq. (82) from Eq. (79), we must use a different set of Hamilton's equations, given by Eq. (72). We see that a Hamiltonian can correspond to more than one set of Hamilton's equations. However, the conventional quantization procedure for the Hamiltonian leads to only a single wave equation, the one corresponding to the conventional Hamilton's equations. We must seek out a different path to quantization.

B. Right way

A standard method of showing the correspondence between quantum and classical mechanics is to start with the Schrödinger wave equation and derive the classical Hamilton-Jacobi equation as an approximation [35]. The procedure is straightforward and can be illustrated for the case of one dimension with coordinate x. Begin with the Schrödinger equation,

$$\left[-\hbar^2\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi=i\hbar\frac{\partial\psi}{\partial t} \tag{83}$$

and write the wave function as

$$\psi(x,t)=A(x,t)\exp\left(\frac{i}{\hbar}S(x,t)\right), \tag{84}$$

where the amplitude and phase are determined by the real functions $A(x,t)$ and $S(x,t)$. The wave equation then becomes

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2 A}{\partial x^2}-\frac{i\hbar}{m}\frac{\partial A}{\partial x}\frac{\partial S}{\partial x}-\frac{i\hbar}{2m}A\frac{\partial^2 S}{\partial x^2}+\frac{1}{2m}A\left(\frac{\partial S}{\partial x}\right)^2 +AV(x)\right]\psi=\left[i\hbar\frac{\partial A}{\partial t}-A\frac{\partial S}{\partial t}\right]\psi. \tag{85}$$

If we separate this expression into real and imaginary parts, we get two equations:

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+V(x)=\frac{\hbar^2}{2mA}\frac{\partial^2 A}{\partial x^2} \tag{86}$$

and

$$m\frac{\partial A}{\partial t}+\frac{\partial A}{\partial x}\frac{\partial S}{\partial x}+\frac{1}{2}A\frac{\partial^2 S}{\partial x^2}=0. \tag{87}$$

Equation (86) is the classical Hamilton-Jacobi equation, with an extra term that is a sort of quantum potential. The term becomes zero for $\hbar=0$, leading to the well-known (and sometimes misleading) observation that quantum mechanics reduces to classical mechanics in the limit as $\hbar$ approaches 0. In this sense, the Hamilton-Jacobi equation is the short-wavelength limit of the Schrödinger equation (Ref. [20], pp. 307–314). Equation (87) is the classical continuity equation with density $\rho=A^2$ and current density $j=(A^2/m)\partial S/\partial x$.

The same procedure can be used to determine the appropriate wave equation corresponding to a fractional-derivative classical system described by Eq. (74). In analogy to Eq. (83), we start by choosing the wave equation

$$\left[-\hbar^2\frac{1}{2m}\frac{\partial^2}{\partial x^2}-i\hbar x_{(1/2)}\frac{\partial}{\partial x_{(-1/2)}}+V(x)-i\frac{1}{2}\gamma x_{(1/2)}^2\right]\psi =i\hbar\frac{\partial\psi}{\partial t}. \tag{88}$$

We need to show that this wave equation reduces to the Hamilton-Jacobi equation in the classical limit. This task can be accomplished by writing the wave function as

$$\psi(x,x_{(1/2)},x_{(-1/2)},t)=A(x,x_{(1/2)},x_{(-1/2)},t) \times\exp\left(\frac{i}{\hbar}S(x,x_{(1/2)},x_{(-1/2)},t)\right). \tag{89}$$

The differentiations can be performed easily since there are no fractional derivatives of the wave function in Eq. (88), only integer-order derivatives with respect to fractional coordinates. The fractional wave equation becomes

$$-\frac{\hbar^2}{2m}\frac{\partial^2 A}{\partial x^2}-i\hbar x_{(1/2)}\frac{\partial A}{\partial x_{(-1/2)}}+Ax_{(1/2)}\frac{\partial S}{\partial x_{(-1/2)}}-\frac{i\hbar}{m}\frac{\partial A}{\partial x}\frac{\partial S}{\partial x} -\frac{i\hbar}{2m}A\frac{\partial^2 S}{\partial x^2}+\frac{1}{2m}A\left(\frac{\partial S}{\partial x}\right)^2+AV(x)-i\frac{1}{2}A\gamma x_{(1/2)}^2 =i\hbar\frac{\partial A}{\partial t}-A\frac{\partial S}{\partial t}, \tag{90}$$

which reduces to the Hamilton Jacobi equation

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+x_{(1/2)}\frac{\partial S}{\partial x_{(-1/2)}}+V(x)-i\frac{1}{2}\gamma x_{(1/2)}^2 =\frac{\hbar^2}{2mA}\frac{\partial^2 A}{\partial x^2}, \tag{91}$$

provided we satisfy the continuity equation

$$m\frac{\partial A}{\partial t}+mx_{(1/2)}\frac{\partial A}{\partial x_{(-1/2)}}+\frac{\partial A}{\partial x}\frac{\partial S}{\partial x}+\frac{1}{2}A\frac{\partial^2 S}{\partial x^2}=0. \tag{92}$$

This procedure demonstrates the consistency between the classical and quantum equations in the limit as $\hbar$ approaches 0, which is the desired result. It is not a derivation of the classical equations from the quantum, since the two classical equations are not necessarily the real and imaginary parts of the wave equation.

The success of the above procedure suggests the following rule: The Hamilton-Jacobi equation

$$H\left(q_{s(i)},\frac{\partial S}{\partial \overline{q_{s(i)}}},t\right)+\frac{\partial S}{\partial t}=0 \tag{93}$$

corresponds to the quantum wave equation

$$\left[H\left(q_{s(i)},-i\hbar\frac{\partial}{\partial \overline{q_{s(i)}}},t\right)\right]\psi=i\hbar\frac{\partial\psi}{\partial t}. \tag{94}$$

This wave equation differs from the incorrect one discussed in Sec. VIII A because differentiation is with respect to the coordinates $\overline{q_{s(i)}}$, rather than the canonical coordinates $q_{s(i)}$.

It may be possible to find a quantum wave equation corresponding to the more complicated form of the Hamilton-Jacobi equation, Eq. (58), which includes fractional time derivatives. However, only integer-order derivatives obey a simple product rule, so there is no simple connection between such a wave equation and the Hamilton-Jacobi equation, as there was for Eq. (90). Moreover, the fractional time derivatives would result in an unsatisfactory wave equation, since there are difficulties defining a positive-definite probability density when time derivatives are not of first order. Even the Klein-Gordon equation allows negative probability densities because of its second-order time derivative [36]. For these reasons, we consider Eq. (94) to be the appropriate wave equation.

IX. PUZZLE

Bateman [13] tells the story of an interesting problem which, at the time, appeared impossible to solve. According to Bateman's account, R. C. Tolman posed the question of whether there were equations that could not be obtained from a Lagrangian. E. T. Whittaker responded with

$$\begin{aligned} \ddot{x}-x&=0, \\ \ddot{y}-\dot{x}&=0, \end{aligned} \tag{95}$$

which he believed might not be derivable from any Lagrangian. In an attempt to solve the problem, Bateman introduced the two tricks described in Sec. I of the present work: the method of dual equations and the time-dependent Lagrangian. Bateman attempted to find a Lagrangian using his methods, but concluded that a solution did not seem possible. To my knowledge, this puzzle has remained unsolved since its publication by Bateman in 1931. However, with fractional mechanics (using the conventions of Sec. VII) it is trivial to find a solution:

$$L=\dot{x}^2+\dot{y}^2+x^2+\dot{x}y-ix_{(1/2)}y_{(1/2)}. \tag{96}$$

X. CONCLUSION

By using fractional derivatives, it is possible to construct a complete mechanical description of nonconservative systems, including Lagrangian and Hamiltonian mechanics, canonical transformations, Hamilton-Jacobi theory, and quantum wave mechanics. The example in Sec. VIII shows that the formalism can be applied to a classical frictional force proportional to velocity. There is no assurance that all nonconservative systems can be treated by these techniques. However, by using fractional derivatives of various orders, it is possible to choose Lagrangians that result in a wide range of dissipative Euler-Lagrange equations. These Lagrangians can describe nonconservative forces involving fractional derivatives, rather than the functions more commonly used to describe dissipation. Hence we are presented with new possibilities for dissipative equations, and also new challenges posed by the complexity of the mathematical methods.

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