In general, a symmetric bilinear form $\beta(x, y)$ is called **nondegenerate** if its **radical** S is 0, where $S = \{x \in L | \beta(x, y) = 0 \text{ for all } y \in L\}$. Because the Killing form is associative, its radical is more than just a subspace: S is an *ideal* of L. From linear algebra, a practical way to test nondegeneracy is as follows: Fix a basis $x_1, \ldots, x_n$ of L. Then κ is nondegenerate if and only if the $n \times n$ matrix whose i, j entry is $\kappa(x_i, x_j)$ has nonzero determinant.

As an example, we compute the Killing form of $\mathfrak{sl}(2, \mathsf{F})$, using the standard basis (Example 2.1), which we write in the order (x, h, y). The matrices become:

$$\text{ad } h = \text{diag}(2, 0, -2), \text{ ad } x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ ad } y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Therefore κ has matrix $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$, with determinant -128, and κ is nondegenerate. (This is still true so long as char $\mathsf{F} \neq 2$.)

Recall that a Lie algebra L is called semisimple in case $\text{Rad} L = 0$. This is equivalent to requiring that L have no nonzero abelian ideals: indeed, any such ideal must be in the radical, and conversely, the radical (if nonzero) includes such an ideal of L, viz., the last nonzero term in the derived series of $\text{Rad} L$ (cf. exercise 3.1).

Theorem. *Let L be a Lie algebra. Then L is semisimple if and only if its Killing form is nondegenerate.*

Proof. Suppose first that $\text{Rad } L = 0$. Let S be the radical of κ. By definition, $Tr(\text{ad } x \text{ ad } y) = 0$ for all $x \in S$, $y \in L$ (in particular, for $y \in [SS]$). According to Cartan's Criterion (4.3), $\text{ad}_L S$ is solvable, hence S is solvable. But we remarked above that S is an ideal of L, so $S \subset \text{Rad } L = 0$, and κ is nondegenerate.

Conversely, let $S = 0$. To prove that L is semisimple, it will suffice to prove that every abelian ideal I of L is included in S. Suppose $x \in I$, $y \in L$. Then ad x ad y maps $L \to L \to I$, and $(\text{ad } x \text{ ad } y)^2$ maps L into $[II] = 0$. This means that ad x ad y is nilpotent, hence that $0 = Tr(\text{ad } x \text{ ad } y) = \kappa(x, y)$, so $I \subset S = 0$. (This half of the proof remains valid even in prime characteristic (Exercise 6).) □

The proof shows that we always have $S \subset \text{Rad } L$; however, the reverse inclusion need not hold (Exercise 4).

5.2. Simple ideals of L

First a definition. A Lie algebra L is said to be the **direct sum** of ideals $I_1, \ldots, I_t$ provided $L = I_1 + \ldots + I_t$ (direct sum of subspaces). This condition forces $[I_i I_j] \subset I_i \cap I_j = 0$ if $i \neq j$ (so the algebra L can be viewed as gotten from the Lie algebras I_i by defining Lie products componentwise for the external direct sum of these as vector spaces). We write $L = I_1 \oplus \ldots \oplus I_t$.