

1.1 Beat Simulations

In order gain some intuitive understanding of the issues surrounding the idea of group velocity [v_g], a set of simulations was created for the following configurations of different beat superposition waves:

1. 1-way, non-dispersive phase waves
2. 2-way, non-dispersive phase waves
3. 1-way, dispersive phase waves
4. 2-way, dispersive phase waves

Each simulation uses the same basic approach in which the amplitude of two individual phase waves is added for all values of [x] and then displayed as a single frame in the animation. The process is then repeated for incrementing values of [t] to create the next frame within the animation. Within each simulation, the superposition wave is calculated using two methods, which act as a verification of the other:

- Direct addition of the phase waves for each value of [x]:

' calculate wave for each frame in time

$$tw1(x) = A0 * \cos((k1 * x) - (w1 * t))$$

$$tw2(x) = A0 * \cos((k2 * x) - (w2 * t))$$

' basic superposition addition

$$sw1(x) = tw1(x) + tw2(x)$$

- As determined by the *'beat wave equation'*:

'See Appendix-A for the derivation of equation below:

$$sw2(x) = 2 * A0 * \cos((kp * x) - (wp * t)) * \cos((kg * x) - (wg * t))$$

In both cases, the two phase waves are given different angular frequencies centred on [ω_0] to create a beat superposition wave, e.g. [$\omega_1 = \omega_0 + \Delta\omega$] and [$\omega_2 = \omega_0 - \Delta\omega$]. The corresponding wave numbers were then calculated based on [$k_1 = \omega_1 / c_1$] and [$k_2 = \omega_2 / c_2$], where [c_1, c_2] correspond to the phase velocity [v_p] of each wave, which are different in the case of a dispersive media. However, in the case of a non-dispersive media, the individual phase waves are assumed to propagate with a phase velocity [$c_1 = c_2 = 1$]. Finally, the direction of the wave propagation [$kx \pm \omega t$] can be change according to the sign of [ωt].

Despite the apparently unambiguous definition of the group velocity [v_g], there is a spread of results that either needs to be corrected or explained. While the actual issues need to be considered in terms of the results of each simulation, it is possible the issues stem from an incorrect formulation of [v_g] used within the simulation:

$$[1] \quad v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{d\omega}{dk}; \quad \begin{array}{ll} v_g < v_p; & \text{if } d\omega < dk \\ v_g > v_p; & \text{if } d\omega > dk \end{array}$$

Based on [1], $[d\omega]$ and $[dk]$ might be expanded to show how $[d\omega]$ could be greater than $[dk]$, when the phase velocity $[v_p < c]$, which might then lead to the idea that $[v_g]$ can be greater than $[v_p]$. However, this would not necessarily explain the 2-way, non-dispersive results.

$$[2] \quad v_g = \frac{\omega_1 - \omega_2}{(\omega_1 / c_1) - (\omega_2 / c_2)} = \frac{d\omega}{dk}$$

In contrast to [1] and [2], the derivation of the mathematical formulation of the beat superposition wave, see Appendix-A, introduced another set of terms that appears to be linked to the idea of group velocity, but now denoted by $[v_G]$:

$$[3] \quad \omega_g = \frac{\omega_1 - \omega_2}{2}; \quad k_g = \frac{k_1 - k_2}{2}; \quad \frac{\omega_g}{k_g} = v_G$$

But what are $[v_g]$ and $[v_G]$ actually describing?

The following tables collate the results of all the animations for comparison. However, it is highlighted that many of the values of $[v_g]$ and $[v_G]$ not only appear to contradict each other, but also the group velocity implied by the animation itself, as reflected by the value $[v]$ in the head of each table:

1-way, non-dispersive, v=+1.00		
$\omega_1=0.22$	$k_1=0.22$	$c_1=+1.00$
$\omega_2=0.18$	$k_2=0.18$	$c_2=+1.00$
$d\omega=0.04$	$dk=0.04$	$v_g=+1.00$
$\omega_g=0.02$	$K_g=0.02$	$V_G=+1.00$

2-way, non-dispersive, v=+10.00		
$\omega_1=0.22$	$k_1=0.22$	$c_1=+1.00$
$\omega_2=0.18$	$k_2=0.18$	$c_2=-1.00$
$d\omega=0.04$	$dk=0.04$	$v_g=+1.00$
$\omega_g=0.2$	$K_g=0.02$	$V_G=+10.00$

The tables above are the 1-way and 2-way variants within a non-dispersive media; while the tables below are the equivalent for a dispersive media. However, the tables below have 2 sets of results to show the reversing of the phase velocity, i.e. c_1, c_2 :

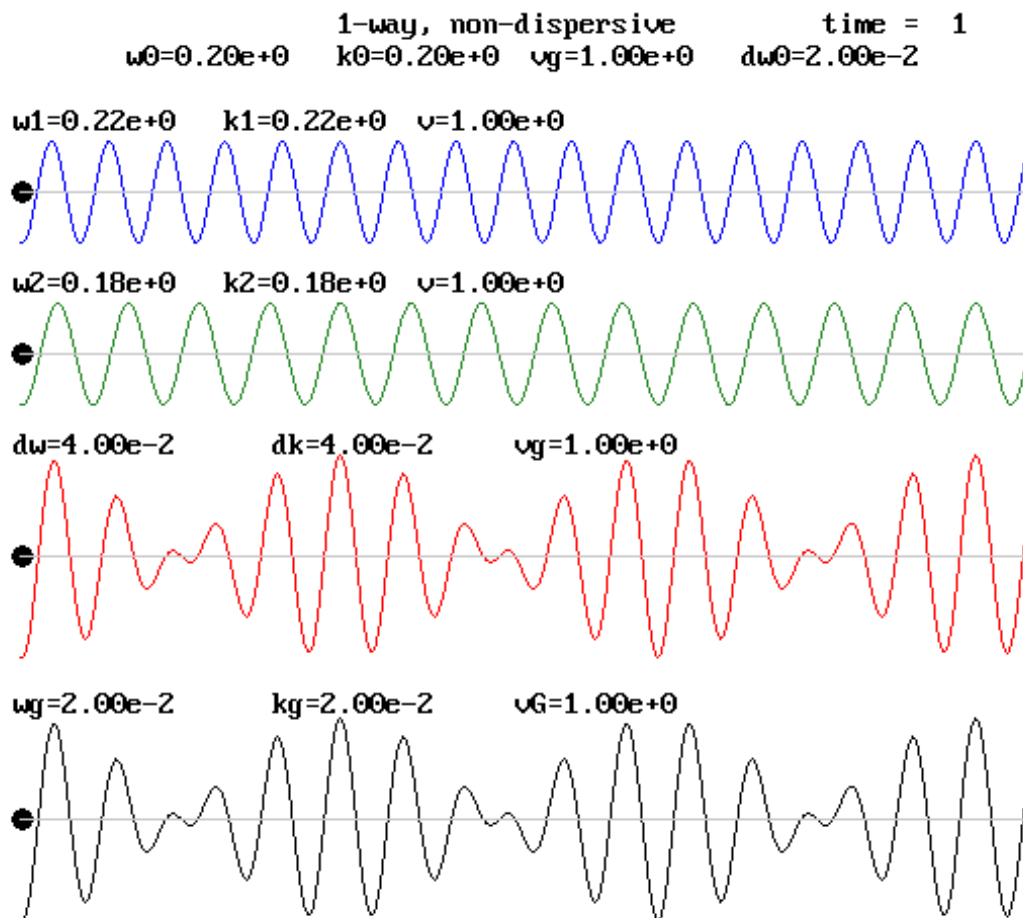
1-way, dispersive, v=+0.925			1-way, dispersive, v=+0.925		
$\omega_1=0.22$	$k_1=0.244$	$c_1=+0.90$	$\omega_1=0.22$	$k_1=0.231$	$c_1=+0.95$
$\omega_2=0.18$	$k_2=0.189$	$c_2=-0.95$	$\omega_2=0.18$	$k_2=0.20$	$c_2=-0.90$
$d\omega=0.04$	$dk=0.055$	$v_g=+0.727$	$d\omega=0.04$	$dk=0.0315$	$v_g=+1.266$
$\omega_g=0.02$	$K_g=0.0275$	$V_G=+0.727$	$\omega_g=0.02$	$K_g=0.0157$	$V_G=+1.266$

1-way, dispersive, $v=+7.27$			1-way, dispersive, $v=+12.66$		
$\omega_1=0.22$	$k_1=0.244$	$c_1=+0.90$	$\omega_1=0.22$	$k_1=0.231$	$c_1=+0.95$
$\omega_2=0.18$	$k_2=0.189$	$c_2=-0.95$	$\omega_2=0.18$	$k_2=0.2$	$c_2=-0.90$
$d\omega=0.04$	$dk=0.055$	$v_g=+0.727$	$d\omega=0.04$	$dk=0.0315$	$v_g=+1.266$
$\omega_g=0.2$	$K_g=0.0275$	$V_G=+7.27$	$\omega_g=0.2$	$K_g=0.0157$	$V_G=+12.66$

At this time, it is proving difficult to reconcile the reason for the spread of results. These results are either an error in the simulation or required some additional explanation of 'how' and 'why' they occur.

1.1.1 1-Way, Non-Dispersive Phase Waves

Within the layout of the animation, the blue-green waves correspond to the 2 phase waves to be added in superposition, as shown by the red wave. The bottom wave in black is generated using the beat wave equation, as defined in Appendix-A.



The black dots on each waveform track the phase velocity [$v_p=c$] in the actual animations, which in the case of a non-dispersive media applies to both the blue-green waves. However, because the blue-green waves are both propagating with the same velocity, and in the same direction, the relationship between these waves remains constant as they propagate

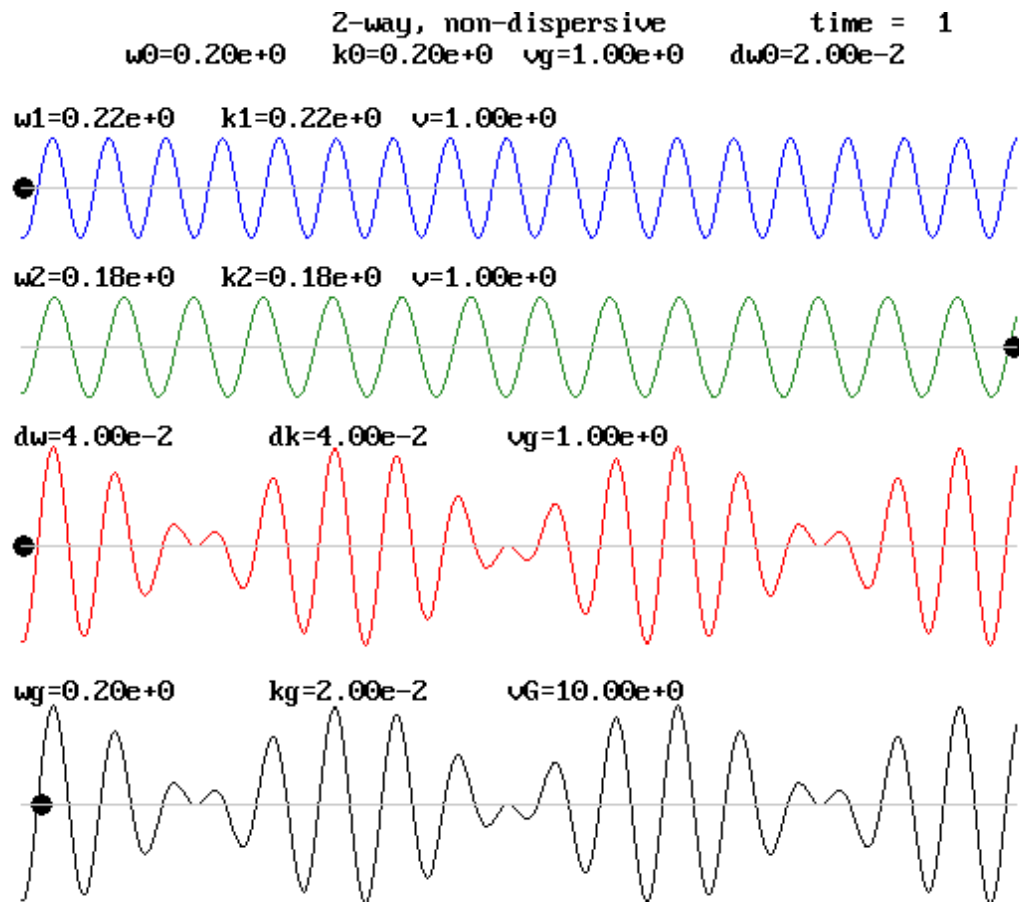
with the same velocity $[c]$. The equality of the beat superposition in red and black appears to reflect the compatibility of the two methods used to calculate the amplitudes. Each wave is annotated with the values of the wave parameters, both assigned and calculated.

What interpretation can be drawn from this first animation?

First of all, it would appear that the beat wave propagates with a velocity $[v_g=v_p]$, which is consistent to the standard description of $[v_g]$. For while the blue-green phase waves have a different frequency and wavelength, they have a common propagation velocity $[v_p=c=1]$, such that the resulting beat waveform remains in-phase with the phase waves as they propagate through space. However, the next simulation suggests a slightly different and confusing picture.

1.1.2 2-Way, Non-Dispersive Phase Waves

While the description of this animation is essentially the same as the *previous case*, the blue-green phase waves are now propagating in opposite directions; although still with the same velocity $[v_p=c]$ as expected in a non-dispersive media.



Despite the change in direction of the green phase wave, there are still some obvious similarities in the beat waveforms, except that the black tracker dot on the axis of the red and black waves now seems to suggest that the beat waveform is '*propagating*' with a much higher velocity than $[v_p]$, i.e. $[>c]$:

Is this suggesting some form of superluminal velocity?

While the answer is assumed to be 'no', the actual explanation is not necessarily obvious. For, at face value, the simulation appears to suggest a conflict between the values shown as the group velocity $[v_g]$, the secondary group velocity $[v_G]$ and the apparent velocity $[v]$ of the beat waveform in the animation.

$$[4] \quad v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{0.04}{0.04} = 1$$

$$v_G = \frac{\omega_1 + \omega_2}{k_1 - k_2} = \frac{0.2}{0.02} = 10 = v$$

While the value of $[v_g=1]$ in [4] appear to be compatible to the idea of $[v_g \leq v_p]$, the value of $[v_G=10]$ appears to correspond to the velocity $[v]$ of the beat waveform in the animation. As far as can be seen, there is no obvious change to the formulation of $[v_g]$ when the direction of the phase waves is reversed, which is not the case for $[v_G]$. However, there is clearly a problem with simply assuming that the beat waveform is actually propagating through space at $[10c]$.

So where is the problem?

Ignoring the obvious possibility of a mistake in the simulation for the moment, consideration might be given to how the animation is produced. Each frame of the animation calculates the value of the amplitudes of the blue-green waves for all values of space $[x]$ at a given point in time $[t]$. Therefore, as the blue-green waves propagate with velocity $[c]$ in opposite directions, the phase relationship between these waves constantly changes with time, i.e. in every frame. So, in this context, the '*shift*' in the beat waveform, which might suggest a propagation velocity greater than $[c]$ has nothing to do with anything propagating through the media, rather it simply reflects the changing phase shift between the blue-green phase waves. However the ambiguity of $[v_g]$ only seems to get worst when trying to simulate the effects of a dispersive media.

1.1.3 1-Way, Dispersive Phase Waves

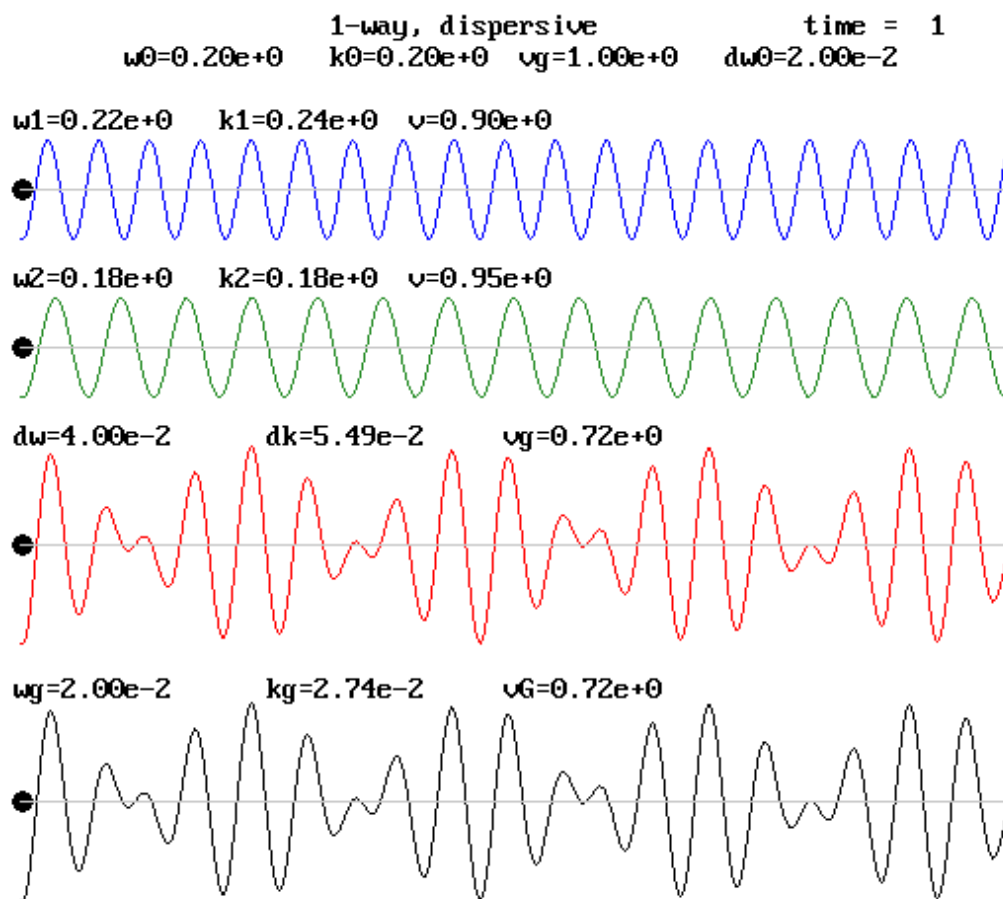
While the mathematical derivation of the group velocity takes into account how $[\omega]$ changes with $[k]$, it was assumed that the effects of a dispersive media might be simulated by simply changing the phase velocity $[v_p=c]$ of each wave, if it is also reflected in the wave number $[k]$, e.g.

[5]

$$c_1 = 0.90; \quad k_1 = \frac{\omega_1}{c_1}$$

$$c_2 = 0.95; \quad k_2 = \frac{\omega_2}{c_2}$$

Based on this assumption, the following animation shows the apparent beat superposition waveform created, when both phase waves are propagating in the same direction, but with different phase velocities:

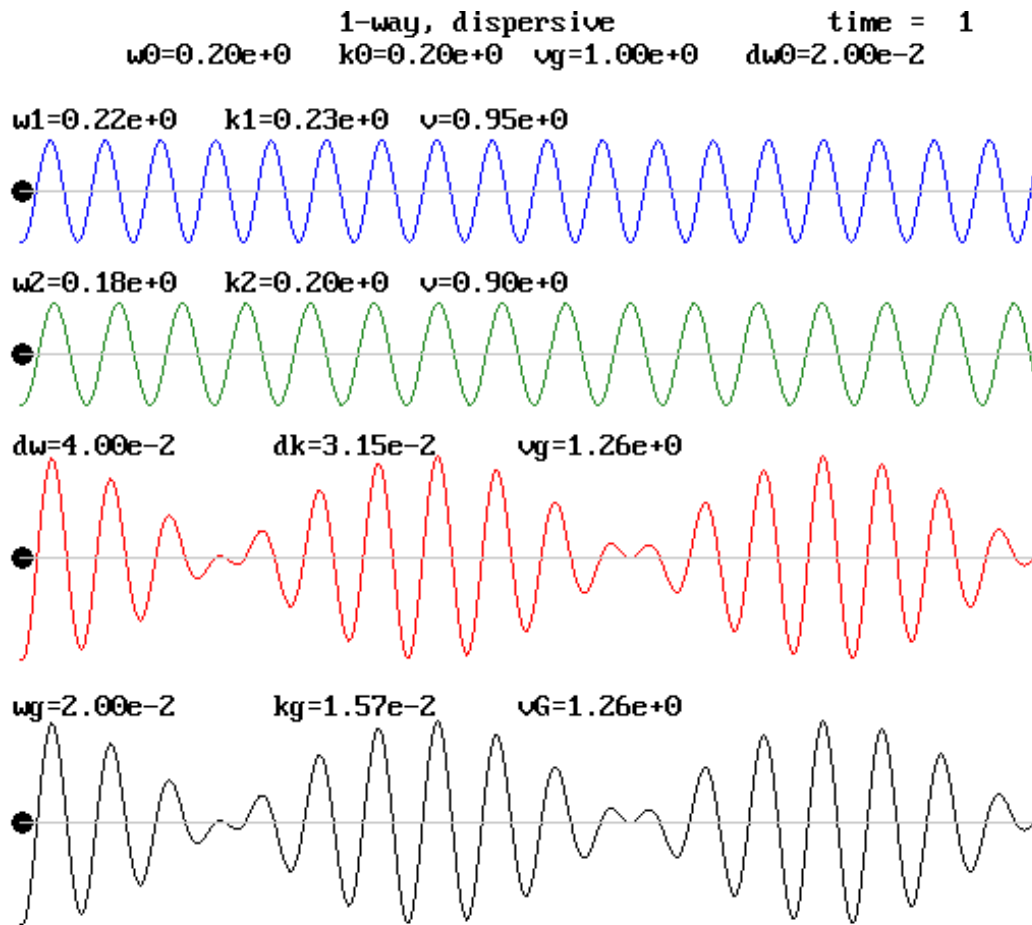


At face value, the animation appears to reflect a similar result to the earlier 1-way, non-dispersive animation, although because the two phase waves are propagating with different velocities, e.g. $[c_1=0.90]$ and $[c_2=0.95]$, the beat waveform continually shifts relative to the phase waves. However, the animation also appears to suggest that the beat waveform is '*propagating*' with a group velocity that is the average of $[c_1+c_2/2=0.925]$ rather than the $[v_g=0.72]$ predicted by $[d\omega/dk]$ or by $[v_g=0.72]$.

What happens the values of $[c_1]$ and $[c_2]$ are reversed?

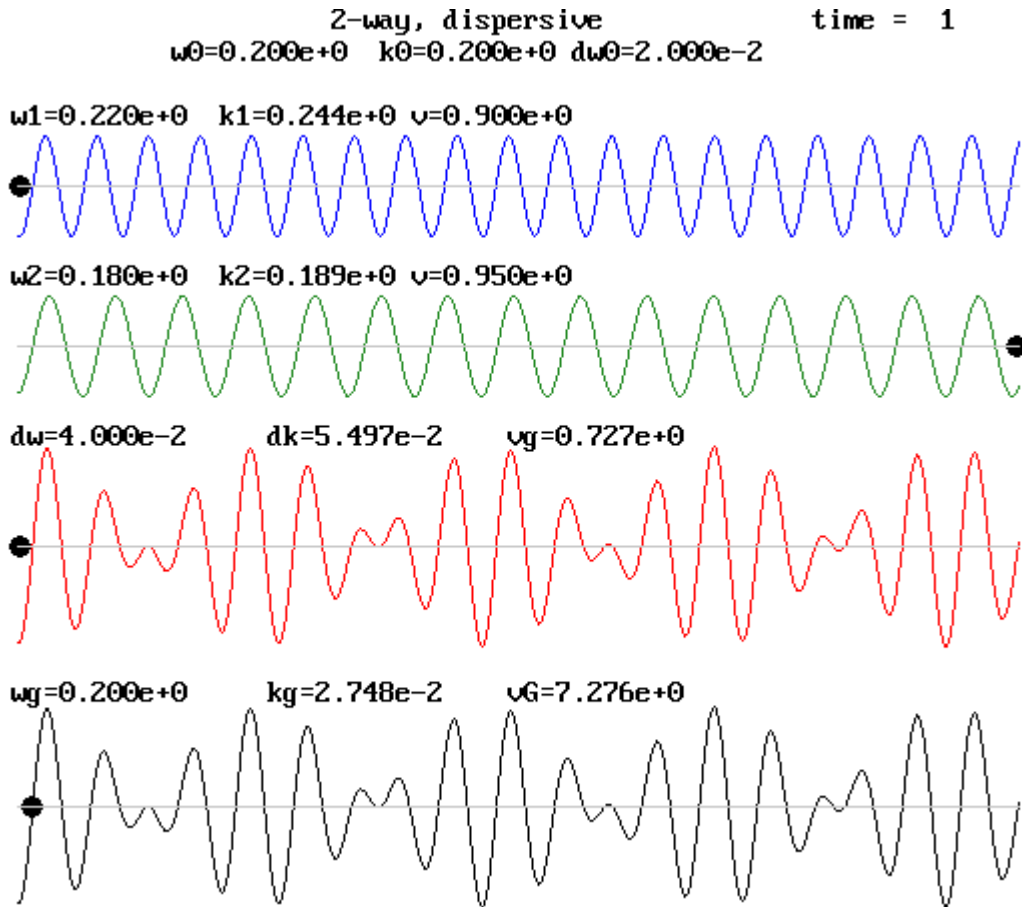
In the following animation, the only change is the reversal of $[c_1=0.95]$ and $[c_2=0.90]$ and while the beat superposition waveform does not appear to change, the values of $[v_g]$ and

$[v_G]$ have now increased above $[v_P]$. Therefore, the results of both animations are questioned.



1.1.4 2-Way, Dispersive Phase Waves

Again, the only thing that changes in this animation is the direction of the second green phase wave. However, as in the '2-way, non-dispersive' animation, the implied group velocity $[v_G=7.276]$ appears to exceed the phase velocity of either phase wave. While the actual value $[v_g=0.726]$ appears identical to the first '1-way, dispersive', where $[c_1=0.90]$ and $c_2=0.95]$, it is still difficult to reconcile the value of $[v_g]$ to the velocity $[v]$ of the beat waveform in the animation.



What happens if we reverse the values of $[c_1]$ and $[c_2]$?

While the animation looks similar to the picture above, the values change, which can be seen in the *earlier comparison tables*.

1.2 Appendix-A: Beat Wave Equations

The idea of a 'beat' wave might initially be explained in terms of sound waves in superposition, where harmonics waves, which are integer multiples of some fundamental frequency, create harmonic beats. However, in order to formulate a mathematical description of these waves, it is assumed that two phase waves have the same amplitude, but different frequencies. Based on [1], different frequencies also require the wavelengths to be different, although in non-dispersive media, all waves are assumed to propagate at the same velocity $[c]$:

$$[1] \quad \lambda = \frac{c}{f}$$

However, unlike standing wave superposition, the underlying phase waves do not have to propagate in opposite directions. To illustrate this fact, the following derivation is based on two phase waves propagating in the same direction.

$$[2] \quad A_1 = A_0 \cos(k_1 x - \omega_1 t); \quad A_2 = A_0 \cos(k_2 x - \omega_2 t)$$

So, as in the description of all superposition waveforms, the process may be illustrated through the addition of just 2 phase wave.

$$[3] \quad A_R = A_1 + A_2 = A_0 [\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)] \equiv A_0 [\cos(A) + \cos(B)]$$

The form of the last expression in [3] is expanded using the following trigonometric identity:

$$[4] \quad \cos(A) + \cos(B) = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

So based on [4], substitute for (A) and (B) as defined by [3]:

$$[5] \quad A_R = 2A_0 \cos\left(\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right) \cos\left(\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right)$$

Initially collecting like-terms together:

$$[6] \quad A_R = 2A_0 \cos\left(\frac{(k_1 + k_2)x}{2} - \frac{(\omega_1 + \omega_2)t}{2}\right) \cos\left(\frac{(k_1 - k_2)x}{2} - \frac{(\omega_1 - \omega_2)t}{2}\right)$$

While different approaches may be taken to simplifying [6], the following substitutions may provide some further insight into the '*nature*' of the group and phase velocities:

$$[7] \quad \begin{aligned} \omega_p &= \frac{\omega_1 + \omega_2}{2}; & k_p &= \frac{k_1 + k_2}{2} \\ \omega_g &= \frac{\omega_1 - \omega_2}{2}; & k_g &= \frac{k_1 - k_2}{2} \Rightarrow \frac{\omega_g}{k_g} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = v_G \end{aligned}$$

So finally, substituting the expression in [7] back into the form of [6]:

$$[8] \quad A_R = 2A_0 \cos(k_p x - \omega_p t) \cos(k_g x - \omega_g t)$$

While, as shown in [7], $[\omega_g/k_g]$ might be seen as an equivalence to $[v_g]$, this equivalence may not be maintained when the waves propagate in opposing directions for the sign within the (B) terms needs to change.