(Note that the operator on the left side is the vector Laplacian, but those on the right have scalar arguments.) Expressions for the vector Laplacian operator in cylindrical and spherical coordinates will be found in the end-paper tables. In those coordinate systems the relationship between the two kinds of Laplacian is much less apparent than in rectangular coordinates, and they are best thought of as quite different operators.

EXERCISE 7.1

Let $\vec{E}(x, y, z, t) = E_o \cos(\omega t - kz)\vec{e}_x$. Find $\vec{\mathbf{E}}$ and $\nabla^2\vec{\mathbf{E}}$.

Answers $\vec{\mathbf{E}} = E_o e^{-jkz}\,\vec{e}_x$. $\nabla^2\vec{\mathbf{E}} = -k^2 E_o e^{-jkz}\,\vec{e}_x$. Note that for this particular $\vec{\mathbf{E}}$, $\nabla^2\vec{\mathbf{E}} = -k^2\vec{\mathbf{E}}$.

In most situations, we shall be dealing with materials that contain no real charge density. This is the case with dielectric materials, and it is also generally the case with conductive materials, because any real charge that may exist will repel itself and thus travel outwards until it resides on the material's outer surfaces. (See Problem 6.5.) Therefore $\nabla \cdot \vec{D} = 0$, and assuming that ϵ is a constant not equal to zero, $\nabla \cdot \vec{E} = 0$. Thus the first term on the right of (7.7) vanishes, and, substituting into (7.6) we obtain

$$\nabla^2\vec{\mathbf{E}} = j\omega\mu(\vec{\mathbf{J}} + j\omega\epsilon\vec{\mathbf{E}}) \tag{7.9}$$

7.3 THE SKIN EFFECT

As our first application of Maxwell's equations, let us investigate the flow of alternating currents in good conductors. We shall see that currents tend to flow on the surface, or "skin." Thus alternating current flow is said to be influenced by the *skin effect*. This effect is of practical importance; it affects resistive losses whenever a high-frequency current flows in an electronic circuit.

Let us assume that the conductive material in question obeys Ohm's law, $\vec{J} = \sigma_E\vec{E}$, where σ_E is the conductivity. Then (7.9) becomes

$$\nabla^2\vec{\mathbf{E}} = j\omega\mu(\sigma_E + j\omega\epsilon)\vec{\mathbf{E}} \tag{7.10}$$

For simplicity, let us now assume that the material in question is a *very good* conductor, so that $\sigma_E \gg |\omega\epsilon|$. (See note[3].) In that case, the displacement-current term, dwarfed by the conduction term, can be neglected, leaving us with $\vec{\nabla}\times\vec{\nabla}\times\tilde{E} = \vec{\nabla}(\vec{\nabla}\cdot\tilde{E}) - \vec{\nabla}^2\tilde{E} = -j\omega\mu\tilde{J} + \omega^2\mu\varepsilon\tilde{E}$

$$\nabla^2\vec{\mathbf{E}} = j\omega\mu\sigma_E\vec{\mathbf{E}} \tag{7.11}$$

Since $\vec{\mathbf{E}} = \vec{\mathbf{J}}/\sigma_E$ we also have

$$\nabla^2\vec{\mathbf{J}} = j\omega\mu\sigma_E\vec{\mathbf{J}} \tag{7.12}$$

$\approx 10^7 \Rightarrow$ 1st TERM DOMINATE

while a similar development (substituting (7.4) into (7.2) to eliminate $\vec{\mathbf{E}}$) results in $\vec{\nabla}\times\vec{\nabla}\times\tilde{H} = \vec{\nabla}(\vec{\nabla}\cdot\tilde{H}) - \vec{\nabla}^2\tilde{H} = -j\omega\sigma\mu\tilde{H} + \omega^2\varepsilon\mu\tilde{H}$

$\vec{\nabla}^2\tilde{H} = j\omega\sigma\mu\tilde{H} - \omega^2\varepsilon\mu\tilde{H}$

$$\nabla^2\vec{\mathbf{H}} = j\omega\mu\sigma_E\vec{\mathbf{H}} \tag{7.13}$$

Equations (7.11)–(7.13) are the basic equations of the skin effect.

To see the significance of these equations, let us consider a conductive material filling the half-space $z < 0$, as shown in Fig. 7.1. Let us imagine that current flows through this material in the x direction, with the current density at the surface being J_o A/m². The current density is independent of y and x; thus, (7.12) simplifies to $\vec{\nabla}^2\tilde{J} = j\omega\mu\sigma\tilde{J}$

RECTANGULAR

$\vec{\nabla}^2\vec{J} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\hat{x}J_x + \hat{y}J_y + \hat{z}J_z)$

$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \Rightarrow \vec{\nabla}^2\vec{J} = \frac{\partial^2\vec{J}}{\partial z^2}$, $J_y = J_z = 0$ $\Rightarrow$

$$\frac{\partial^2\mathbf{J}_x}{\partial z^2} = j\omega\mu\sigma_E\mathbf{J}_x \tag{7.14}$$

which has the solution

$$\mathbf{J}_x = Ae^{-(1+j)z/\delta} + Be^{(1+j)z/\delta} \tag{7.15}$$

where A and B are any constants, and δ, given by

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_E}} \tag{7.16}$$

is known as the *skin depth*.

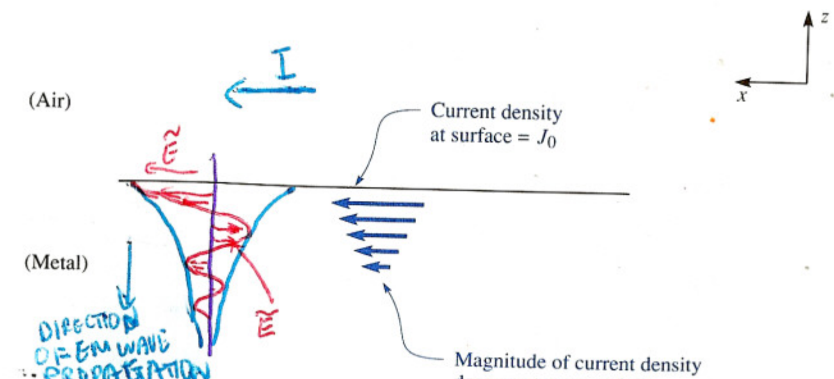

Figure 7.1 Skin effect at the surface of an imperfectly conducting metal.

[3]The absolute-value bars are used because for some materials the value of ϵ can be negative. What matters is that the term containing $\omega\epsilon$ should be negligible compared with the term containing σ_E.