VII.7 *SO*(10) Unification

Each family into a single representation

At the end of chapter VII.5 we felt we had good reason to think that $SU(5)$ unification is not the end of the story. Let us ask if we might be able to fit the 5 and 10^* into a single representation of a bigger group G containing $SU(5)$.

It turns out that there is a natural embedding of $SU(5)$ into the orthogonal $SO(10)$ that works,[1] but to explain that I have to teach you some group theory. The starting point is perhaps somewhat surprising: We go back to chapter II.3, where we learned that the Lorentz group $SO(3, 1)$, or its Euclidean cousin $SO(4)$, has spinor representations. We will now generalize the concept of spinors to d-dimensional Euclidean space. I will work out the details for d even and leave the odd dimensions as an exercise for you. You might also want to review appendix B now.

Clifford algebra and spinor representations

Start with an assertion. For any integer n we claim that we can find $2n$ hermitean matrices γ_i $(i = 1, 2, \cdots, 2n)$ that satisfy the Clifford algebra

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \tag{1}$$

In other words, to prove our claim we have to produce $2n$ hermitean matrices γ_i that anticommute with each other and square to the identity matrix. We will refer to the γ_i's as the γ matrices for $SO(2n)$.

For $n = 1$, it is a breeze: $\gamma_1 = \tau_1$ and $\gamma_2 = \tau_2$. There you are.

[1] Howard Georgi told me that he actually found $SO(10)$ before $SU(5)$.

Now iterate. Given the $2n$ γ matrices for $SO(2n)$ we construct the $(2n+2)$ γ matrices for $SO(2n+2)$ as follows

$$\gamma_j^{(n+1)} = \gamma_j^{(n)} \otimes \tau_3 = \begin{pmatrix} \gamma_j^{(n)} & 0 \\ 0 & -\gamma_j^{(n)} \end{pmatrix}, \quad j = 1, 2, \cdots, 2n \tag{2}$$

$$\gamma_{2n+1}^{(n+1)} = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

$$\gamma_{2n+2}^{(n+1)} = 1 \otimes \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4}$$

(Throughout this book 1 denotes a unit matrix of the appropriate size.) The superscript in parentheses is obviously for us to keep track of which set of γ matrices we are talking about. Verify that if the $\gamma^{(n)}$'s satisfy the Clifford algebra, the $\gamma^{(n+1)}$'s do as well. For example,

$$\begin{aligned}\{\gamma_j^{(n+1)}, \gamma_{2n+1}^{(n+1)}\} &= (\gamma_j^{(n)} \otimes \tau_3) \cdot (1 \otimes \tau_1) + (1 \otimes \tau_1) \cdot (\gamma_j^{(n)} \otimes \tau_3) \\ &= \gamma_j^{(n)} \otimes \{\tau_3, \tau_1\} = 0\end{aligned}$$

This iterative construction yields for $SO(2n)$ the γ matrices

$$\gamma_{2k-1} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \tau_1 \otimes \tau_3 \otimes \tau_3 \otimes \cdots \otimes \tau_3 \tag{5}$$

and

$$\gamma_{2k} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \tau_2 \otimes \tau_3 \otimes \tau_3 \otimes \cdots \otimes \tau_3 \tag{6}$$

with 1 appearing $k-1$ times and τ_3 appearing $n-k$ times. The γ's are evidently 2^n by 2^n matrices. When and if you feel confused at any point in this discussion you should work things out explicitly for $SO(4)$, $SO(6)$, and so on.

In analogy with the Lorentz group, we define $2n(2n-1)/2 = n(2n-1)$ hermitean matrices

$$\sigma_{ij} \equiv \frac{i}{2}[\gamma_i, \gamma_j] \tag{7}$$

Note that σ_{ij} is equal to $i\gamma_i\gamma_j$ for $i \neq j$ and vanishes for $i = j$. The commutation of the σ's with each other is thus easy to work out. For example,

$$[\sigma_{12}, \sigma_{23}] = -[\gamma_1\gamma_2, \gamma_2\gamma_3] = -\gamma_1\gamma_2\gamma_2\gamma_3 + \gamma_2\gamma_3\gamma_1\gamma_2 = -[\gamma_1, \gamma_3] = 2i\sigma_{13}$$

Roughly speaking, the γ_2's in σ_{12} and σ_{23} knock each other out. Thus, you see that the $\frac{1}{2}\sigma_{ij}$'s satisfy the same commutation relations as the generators J^{ij}'s of $SO(2n)$ (as given in appendix B). The $\frac{1}{2}\sigma_{ij}$'s represent the J^{ij}'s.

As 2^n by 2^n matrices, the σ's act on an object ψ with 2^n components that we will call the spinor ψ. Consider the unitary transformation $\psi \to e^{i\omega_{ij}\sigma_{ij}}\psi$ with $\omega_{ij} = -\omega_{ji}$ a set of real numbers. Then

$$\psi^\dagger\gamma_k\psi \to \psi^\dagger e^{-i\omega_{ij}\sigma_{ij}}\gamma_k e^{i\omega_{ij}\sigma_{ij}}\psi = \psi^\dagger\gamma_k\psi - i\omega_{ij}\psi^\dagger[\sigma_{ij}, \gamma_k]\psi + \cdots$$

for ω_{ij} infinitesimal. Using the Clifford algebra we easily evaluate the commutator as $[\sigma_{ij}, \gamma_k] = -2i(\delta_{ik}\gamma_j - \delta_{jk}\gamma_i)$. (If k is not equal to either i or j then γ_k clearly commutes with σ_{ij}, and if k is equal to either i or j, then we use $\gamma_k^2 = 1$.) We see that the set of objects $v_k \equiv \psi^\dagger \gamma_k \psi$, $k = 1, \cdots, 2n$ transforms as a vector in $2n$-dimensional space, with $4\omega_{ij}$ the infinitesimal rotation angle in the ij plane:

$$v_k \to v_k - 2(\omega_{kj} v_j - \omega_{ik} v_i) = v_k - 4\omega_{kj} v_j \tag{8}$$

(in complete analogy to $\bar{\psi}\gamma^\mu\psi$ transforming as a vector under the Lorentz group.) This gives an alternative proof that $\frac{1}{2}\sigma_{ij}$ represents the generators of $SO(2n)$.

We define the matrix $\gamma^{\text{FIVE}} = (-i)^n \gamma_1\gamma_2 \cdots \gamma_{2n}$, which in the basis we are using has the explicit form

$$\gamma^{\text{FIVE}} = \tau_3 \otimes \tau_3 \otimes \cdots \otimes \tau_3 \tag{9}$$

with τ_3 appearing n times. By analogy with the Lorentz group we define the "left handed" spinor $\psi_L \equiv \frac{1}{2}(1 - \gamma^{\text{FIVE}})\psi$ and the "right handed" spinor $\psi_R \equiv \frac{1}{2}(1 + \gamma^{\text{FIVE}})\psi$, such that $\gamma^{\text{FIVE}}\psi_L = -\psi_L$ and $\gamma^{\text{FIVE}}\psi_R = \psi_R$. Under $\psi \to e^{i\omega_{ij}\sigma_{ij}}\psi$, we have $\psi_L \to e^{i\omega_{ij}\sigma_{ij}}\psi_L$ and $\psi_R \to e^{i\omega_{ij}\sigma_{ij}}\psi_R$ since γ^{FIVE} commutes with σ_{ij}. The projection into left and right handed spinors cut the number of components into halves and thus we arrive at the important conclusion that the two irreducible spinor representations of $SO(2n)$ have dimension 2^{n-1}. (Convince yourself that the representation cannot be reduced further.) In particular, the spinor representation of $SO(10)$ is $2^{10/2-1} = 2^4 = 16$—dimensional. We will see that the 5^* and 10 of $SU(5)$ can be fit into the 16 of $SO(10)$.

Embedding unitary groups into orthogonal groups

The unitary group $SU(5)$ can be naturally embedded into the orthogonal group $SO(10)$. In fact, I will now show you that embedding $SU(n)$ into $SO(2n)$ is as easy as $z = x + iy$.

Consider the $2n$-dimensional real vectors $x = (x_1, \cdots, x_n, y_1, \cdots, y_n)$ and $x' = (x'_1, \cdots, x'_n, y'_1, \cdots, y'_n)$. By definition, $SO(2n)$ consists of linear transformations on these two real vectors leaving their scalar product $x'x = \sum_{j=1}^n (x'_j x_j + y'_j y_j)$ invariant.

Now out of these two real vectors we can construct two n-dimensional complex vectors $z = (x_1 + iy_1, \cdots, x_n + iy_n)$ and $z' = (x'_1 + iy'_1, \cdots, x'_n + iy'_n)$. The group $U(n)$ consists of transformations on the two n-dimensional complex vectors z and z' leaving invariant their scalar product

$$\begin{aligned}(z')^* z &= \sum_{j=1}^n (x'_j + iy'_j)^*(x_j + iy_j) \\ &= \sum_{j=1}^n (x'_j x_j + y'_j y_j) + i\sum_{j=1}^n (x'_j y_j - y'_j x_j)\end{aligned}$$

In other words, $SO(2n)$ leaves $\sum_{j=1}^n (x'_j x_j + y'_j y_j)$ invariant, but $U(n)$ consists of the subset of those transformations in $SO(2n)$ that leave invariant not only $\sum_{j=1}^n (x'_j x_j + y'_j y_j)$ but also $\sum_{j=1}^n (x'_j y_j - y'_j x_j)$.

Now that we understand this natural embedding of $U(n)$ into $SO(2n)$, we see that the defining or vector representation of $SO(2n)$, which we will call simply $2n$, decomposes