Stürm-Liouville Problems

THEOREM: Consider the *Stürm-Liouville problem:*

$$p(x)\, y'' + p'(x)\, y' + q(x)\, y = k\, y$$

on the interval $[a, b]$ with $p(x)$ and $q(x)$ continuous, and $p(x) > 0$. Assume either classical separated boundary conditions, namely, one of:

$$y(x) = 0, \qquad y'(x) = 0, \qquad y'(x) = c\, y(x)$$

at each endpoint $x = a$ and $x = b$; or else periodic boundary conditions:

$$y(a) = y(b), \qquad y'(a) = y'(b).$$

Then:

(a) the only solutions are for k real.

(b) the set of all linearly independent solutions is complete for the usual space of functions.

(c) the set of linearly independent solutions is orthogonal.

COROLLARY: If the condition $p(x) > 0$ fails because, at one endpoint, $p(x) = 0$, then the classical separated boundary condition at that endpoint is replaced by:

$$y(x) < \infty$$

at that endpoint.

COROLLARY: If the *Stürm-Liouville problem* is modified by:

$$p(x)\, y'' + p'(x)\, y' + q(x)\, y = k\, r(x)\, y$$

with $r(x)$ positive and continuous, and all other conditions the same, then the conclusions of the Theorem are still true, except that orthogonality in conclusion (c) is replaced by a "weighted orthogonality" with weighting function $r(x)$:

$$\int_a^b y_n(x)\, y_m(x)\, r(x)\, dx = 0 \quad \text{whenever} \quad n \neq m.$$

Example 1: $y'' = ky$ on $[0, \ell]$ with $y(0) = 0$, $y(\ell) = 0$. Then there are solutions only for $k = -\lambda_n^2 = -\dfrac{n^2\pi^2}{\ell^2}$, $n = 1, 2, 3, \ldots$ and $y_n = c_n \sin\left(\dfrac{n\pi x}{\ell}\right)$. This gives the Sine series.

Example 2: $y'' = ky$ on $[0, \ell]$ with $y'(0) = 0$, $y'(\ell) = 0$. Then there are solutions for $k = -\lambda_n^2 = -\dfrac{n^2\pi^2}{\ell^2}$, $n = 1, 2, 3, \ldots$ and for $k = 0$. In the second case, $y = 1$ and for $n = 1, 2, 3, \ldots,\ y_n = c_n \cos\left(\dfrac{n\pi x}{\ell}\right)$. This gives the Cosine series.

Example 3: $y'' = ky$ on $[a, b]$ with $y(a) = y(b)$, $y'(a) = y'(b)$. Then there are solutions for $k = -\lambda_n^2 = -\dfrac{4n^2\pi^2}{(b-a)^2}$, $n = 1, 2, 3, \ldots$ as well as $k = 0$. Further, for each $n \geq 1$ there are two linearly independent solutions, $y_n = a_n \sin\left(\dfrac{2n\pi x}{b-a}\right)$ and $y_n = b_n \cos\left(\dfrac{2n\pi x}{b-a}\right)$. For $k = 0$, $y_0 = a_0$. This gives the Fourier series.

Example 4: $xy'' + y' = kxy$ with $y(0) < \infty$ and $y(\ell) = 0$. There are solutions for certain $k_n = -\lambda_n^2 < 0$, $n = 1, 2, 3, \ldots$ Call the solutions $g_n(x)$. Then the functions $g_n(x)$ are complete on the interval $[0, \ell]$ and are orthogonal with respect to the weight x:

$$\int_0^\ell g_m(x)\, g_n(x)\, x\, dx = 0 \quad \text{whenever} \quad n \neq m.$$

It turns out that $g_n(x) = J_0(\lambda_n x)$ for the zeroth Bessel function $J_0(x)$.