T_0 - and T_1 -Spaces

Sources:
Munkres = "Topology", 2nd edition, 2004
Schaum's = "General Topology" by Lipschutz, 1st edition, 1965
Milewski = "Topology Problem Solvers", 1st edition, 1998
Gemignani = "Elementary Topology", 2nd edition, 1972
Willard = "General Topology", 2004 edition
Conover = "A First Course in Topology," 1975 edition
Adamson = "A General Topology Workbook," 1st edition, 1996
Ponomarev = "Fundamentals of General Topology: Problems and Exercises", 1984 edition

Definition: A topological space *X* is said to be T_0 (or a *Kolmogorov space*) if given any two distinct points $x, y \in X$, there is an open subset of *X* containing least one point which does not contain the other. A topological space *X* is said to be T_1 (or a *Frechet space*) if given any two distinct points $x, y \in X$, there is an open subset of *X* containing *x* which does not contain *y* and an open subset of *X* containing *y* which does not contain *x*.

Well-Known Theorems:

- Every T_1-space is a T_0-space. (proven)
- A topological space *X* is T_0 if and only if given any two distinct points *x* and *y* of *X*, either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. (proven)
- A topological space *X* is T_0 if and only if distinct one-point subsets of *X* have distinct closures. (proven)
- The property of being a T_0-space is a topological property. (proven)
- The property of being a T_1-space is a topological property. (proven)
- A topological space is T_1 if and only if every single-point set is closed. (proven)
- A topological space is a T_1-space if and only if every subset is the intersection of all the open sets containing it. (proven)
- A subspace of a T_0-space is T_0. (proven)
- A subspace of a T_1-space is T_1. (proven)
- A product of nonempty T_0-spaces is T_0 if and only if each component space is T_0. (proven)
- A product of nonempty T_1-spaces is T_1 if and only if each component space is T_1. (proven)
- Every metric space is T_1 (and therefore T_0). (proven)
- If *A* is a subset of a T_1-space *X*, then *x* is a limit point of *A* if and only if every open set containing *x* contains infinitely many points of *A*. (proven)
- If *A* be a subset of a T_1-space *X*, and $x \in \overline{A}$ with $x \notin A$, then every open subset of *X* containing *x* contains infinitely many points of *A*. (proven)
- The only topology on *X* which makes *X* into a T_1-space is the discrete topology. (proven)
- If *X* is a T_1-space and $\mathcal{B}_x$ is a local basis at $x \in X$, then for every $y \in X$ distinct from *x*, there is some member of $\mathcal{B}_x$ that does not contain *x*. (proven)
- If *X* is a first countable T_1-space, and *A* is a subset of *X*, and $p \in X$ is a limit point of *A*, then there exists a sequence of distinct terms in *A* that converges to *p*. (proven)
- A pseudometric space (X, d) is a metric space if and only if it is a T_0-space. (proven)
- A quasimetric space is a T_1-space. (proven)
- A T_1-space with a finite basis for its topology is finite and has the discrete topology. (proven)

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Minor Theorems:

- Given a space *X*, define $\sim$ by $x \sim y$ iff $\overline{\{x\}} = \overline{\{y\}}$. Then the resulting quotient space $X/\sim$ is T_0. (proven)
- A subset *A* of a T_1-space is countably compact if and only if every countable open cover of *A* has a finite subcover. (proven)
- If *X* has any particular point topology, then *X* is a T_0-space. (proven)
- Let *X* be a topological space and define a relation *R* on *X* by setting $(x, y) \in R$ if and only if $x \in \overline{\{y\}}$. Then *R* is reflexive and transitive, and *R* is a partial order relation if and only if *X* is a T_0-space. (proven)
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Major Theorems:

1. (Conover, p.162) Prove that every T_1-space is a T_0-space.

Let *X* be a T_1-space. Let $x, y \in X$ be two distinct points. There is an open subset of *X* containing *x* which does not contain *y* and an open subset of *X* containing *y* which does not contain *x*. In particular, there is an open subset of *X* containing least one point which does not contain the other. ■

2. (Gemignani, p.92) Give an example of a space that is not T_0, and an example of a T_0-space that is not T_1.

Let *X* be any set with more than one points. Give *X* the trivial topology, i.e. $\{X, \varnothing\}$. Then for any distinct points $x, y \in X$, the only open set containing either point is *X* and so there is no way to separate *x* from *y* or *y* from *x*. Thus *X* is not T_0.

Another example of a non T_0-space is a *pseudometric space* (where a pseudometric is a function satisfying the axioms of a metric except two distinct points may have zero distance). Let *x* and *y* be two distinct points of a pseudometric space (X, d) such that $d(a,b) = 0$. Then since any *d*-open ball (and therefore any neighbourhood) of *a* contains *b* and any *d*-open ball (and therefore any neighbourhood) of *b* contains *a*. Thus *a* and *b* cannot be separated and so *X* is not T_0.

Let $X = \{x, y\}$ and *X* is given the topology $\{X, \{x\}, \varnothing\}$, then *X* is a T_0-space since x, y are the only pair of distinct points in *X* and $x \in \{x\}$ and $\{x\}$ does not contain *y*, but *X* is not a T_1-space since there is no open set containing *y* that does not contain *x*.

Another example a T_0-space that is not T_1 is a topological space with a particular point topology (see problem 24). ■

3. (Gemignani, p.92) Prove that if *x* and *y* are two distinct points of a topological space *X*, then every open set which contains either *x* or *y* contains both if and only if

$$\overline{\{x\}} = \overline{\{y\}}.$$

$\Rightarrow$ Suppose *x* and *y* are two distinct points of a topological space *X* and every open set which contains either *x* or *y* contains both. Let $z \in \overline{\{x\}}$. Then every open subset *U* of *X* containing *z* intersects $\{x\}$, i.e. contains *x*. But $x \in \overline{\{x\}} = \overline{\{y\}}$, so that *U* containing *x* means that *U* intersects $\{y\}$. Thus every open set containing *z* intersects $\{y\}$, and thus $z \in \overline{\{y\}}$. Thus $\overline{\{x\}} \subset \overline{\{y\}}$. By the symmetric roles of *x* and *y*, we also have $\overline{\{y\}} \subset \overline{\{x\}}$. Thus $\overline{\{x\}} = \overline{\{y\}}$.

$\Leftarrow$ Suppose $\overline{\{x\}} = \overline{\{y\}}$. Since $x \in \overline{\{x\}} = \overline{\{y\}}$, then every open subset of *X* containing *x* intersects $\{y\}$, i.e. contains *y*, and similarly, $y \in \overline{\{y\}} = \overline{\{x\}}$ so that every open subset of *X* containing *y* contains *x*. ∎

4. (Gemignani, p.92) Prove a topological space *X* is T_0 if and only if given any two distinct points *x* and *y* of *X*, either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

$\Rightarrow$ Suppose *X* is T_0. Let *x* and *y* be two distinct points of *X*. Assume that $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Then $\overline{\{x\}} \subset \overline{\overline{\{y\}}} = \overline{\{y\}}$ and $\overline{\{y\}} \subset \overline{\overline{\{x\}}} = \overline{\{x\}}$, and so $\overline{\{x\}} = \overline{\{y\}}$. By problem 3, every open set which contains either *x* or *y* contains both *x* and *y*, contradicting the assumption that *X* is T_0. Thus we must have either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

$\Leftarrow$ Suppose that given any two distinct points *x* and *y* of *X*, either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Since $x \in \overline{\{x\}}$ and $y \in \overline{\{y\}}$, then we have $\overline{\{x\}} \neq \overline{\{y\}}$. Thus by Proposition 1, there exists an open set that contains *x* and not *y* (or vice versa). Thus *X* is T_0. ∎

5. (Gemignani, p.94 #8) Prove a topological space *X* is T_0 if and only if distinct one-point subsets of *X* have distinct closures.

$\Rightarrow$ Suppose *X* is T_0. Let *x* and *y* be two distinct points of *X*. Suppose $\overline{\{x\}} = \overline{\{y\}}$. Then by Proposition 1, every open set which contains either *x* or *y* contains both *x* and *y*, contradicting the assumption that *X* is T_0. Thus we must have $\overline{\{x\}} \neq \overline{\{y\}}$.

$\Leftarrow$ Suppose that given any two distinct points *x* and *y* of *X*, we have $\overline{\{x\}} \neq \overline{\{y\}}$. Then by problem 3, there exists an open set that contains *x* and not *y*, or an open set that contains *y* and not *x*. Thus *X* is T_0. ∎

6. (Schaum's, p.143) Prove that a topological space *X* is T_1 if and only if every finite subset of *X* is closed.

$\Rightarrow$ Suppose *X* is a T_1-space. We shall prove that every single-point set is closed, or that equivalently its complement is open, and the result will follow. Let $x \in X$ and let $y \in X - \{x\}$. Since *X* is a T_1-space, and $x \neq y$, then there exists an open set U_y such that $y \in U_y$ but $x \notin U_y$. Consequently,

$$X - \{x\} = \bigcup_{y \in X - \{x\}} U_y$$

is a union of open sets and hence is open. Thus $\{x\}$ is closed. Consequently, every finite subset of *X* is a finite union of closed sets and hence is closed.

Gemignani proves that $\{x\}$ is closed by proving that $\{x\}=\overline{\{x\}}$: Firstly, $\{x\}\subset\overline{\{x\}}$ by definition of closure. Let $y\in\overline{\{x\}}$. Then every open set containing *y* intersects $\{x\}$, i.e. contains *x*. Thus we must have $y=x$, otherwise *x* and *y* could not be separated by open sets, contradicting that *X* is T_1. Thus $\overline{\{x\}}\subset\{x\}$, so $\{x\}=\overline{\{x\}}$, and thus $\{x\}$ is closed.

$\Leftarrow$ Suppose every finite subset of *X* is closed. Then in particular every single-point set is closed. Let *x* and *y* be two distinct points in *X*. Then $X-\{x\}$ is an open set containing *y* but not *x*, and $Y-\{y\}$ is an open set containing *x* but not *y*. Thus *X* is a T_1-space.

Gemignani uses closures again: Suppose every single-point set is closed. Then $\{x\}=\overline{\{x\}}$ for all $x\in X$. Let *x* and *y* be two distinct points in *X*. If every open set containing *x* also contains *y*, i.e. intersects $\{y\}$, then $x\in\overline{\{y\}}$. But $\overline{\{y\}}=\{y\}$, so $x=y$, a contradiction. Thus there must be an open set containing *x* but not *y*. Similarly, there must be an open set containing *y* but not *x*. Thus *X* is a T_1-space. ■

7. (Willard, p.86) Prove that *X* is a T_1-space if and only if every subset of *X* is the intersection of all the open sets containing it.

$\Rightarrow$ Suppose *X* is a T_1-space. Let *A* be a subset of *X*. Then $X-A=\bigcup_{x\notin A}\{x\}$, and so by DeMorgan's Law, we have

$$A=\bigcap_{x\notin A}(X-\{x\}),$$

which precisely the intersection of all sets containing. Furthermore, since *X* is a T_1-space, then every single-point set is closed by problem 6, and thus the complements of single-point sets are closed. Thus $A=\bigcap_{x\notin A}(X-\{x\})$ is the intersection of the open sets containing it.

$\Leftarrow$ Suppose every subset of *X* is the intersection of all the open sets containing it. Let $x\in X$. Then $\{x\}$ is the intersection of all the open sets containing *x*. Let $y\in X$ be any point distinct from *x*. If every open set containing *x* also contains *y*, then the intersection of all those open sets would contain $\{x,y\}$, contradicting the assumption that the intersection of all the open sets containing *x* is $\{x\}$. Thus there must be an open set containing *x* but not *y*. Similarly, by the symmetric roles of *x* and *y*, we conclude that there must be an open set containing *y* but not *x*. Thus *X* is a T_1-space. ■

8. (Milewski, p.504) Prove that the property of being a T_1-space [T_0-space] is a topological property, that is, it is invariant under homeomorphisms.

Let *X* be a T_1-space [T_0-space], and let $f:X\to Y$ be a homeomorphism. Let $x,y\in Y$ be two distinct points in *Y*. Then there exist $a,b\in X$ such that $f(a)=x$ and $f(b)\in y$. Then there exist open subsets *U* and *V* of *X* such that $a\in U, b\notin U$ and [or] $b\in V, a\notin V$. Since *f* is an open map, then $f(U)$ is an open subset of *Y* containing $f(a)=x$ and [or] $f(V)$ is an open subset of *Y* containing $f(b)=y$. Furthermore, since *f* is bijective, then $y=f(b)\notin f(U)$ and [or] $x=f(a)\notin f(V)$. Thus *Y* is a T_1-space [T_0-space]. ■

Note that continuity of *f* is not enough (nor needed) to ensure that *Y* is a T_1-space [T_0-space]. Only the surjectivity of *f* and the continuity of f^{-1} are needed.

Milewski's proof that being T_1 is a topological invariant:

Let X be a T_1-space, and let $f : X \to Y$ be a homeomorphism. Let $y \in Y$. Since f is bijective, then $f^{-1}(\{y\})$ is a one-point set, and therefore a closed subset of X by problem 6. The surjectivity of f means that $f(f^{-1}(\{y\})) = \{y\}$, which is a closed subset of Y since f is a homeomorphism and hence a closed map. Thus all single-point sets in Y are closed and so Y is a T_1-space. ■

9. (Gemignani, p.94 #6b) Prove that a subspace of a T_0-space is T_0.

Let X be a T_0-space and let Y be a subspace of X. Let a and b be two distinct points in Y. Since X is a T_0-space, then there exists an open set U such that $a \in U, b \notin U$ or an open set V such that $b \in V, a \notin V$. Consequently, we have either $a \in U \cap Y, b \notin U \cap Y$ or $b \in V \cap Y, a \notin V \cap Y$. Since $U \cap Y$ and $V \cap Y$ are open subsets of Y in its subspace topology, then consequently Y is a T_0-space. ■

10. (Schaum's, p.143) Prove that a subspace of a T_1-space is T_1.

First solution (by me):

Let X be a T_1-space and let Y be a subspace of X. Let a and b be two distinct points in Y. Since X is a T_1-space, then there exists open sets U and V such that $a \in U, b \notin U$ and $b \in V, a \notin V$. Consequently, we have $a \in U \cap Y, b \notin U \cap Y$ and $b \in V \cap Y, a \notin V \cap Y$. Since $U \cap Y$ and $V \cap Y$ are open subsets of Y in its subspace topology, then consequently Y is a T_1-space. ■

Alternate solution (by Schaum's):

Let X be a T_1-space and let Y be a subspace of X. Let $y \in Y$. By problem 6, $X - \{y\}$ is a closed subset of X. Then $(X - \{y\}) \cap Y$ is a closed subset of Y (in its subspace topology). But

$$(X - \{y\}) \cap Y = X \cap Y - \{y\} \cap Y = Y - \{y\}.$$

Thus $Y - \{y\}$ is a closed subset of Y and so Y is a T_1-space. ■

11. (Gemignani, p.94 #6c) Prove that a product of nonempty T_1-spaces [T_0-spaces] is T_1 [T_0] if and only if each component space is T_1 [T_0].

$\Rightarrow$ Suppose $\prod_{i \in I} X_i$ is T_1 [T_0] (where $\prod_{i \in I} X_i$ is given either the product topology or box topology). Given $i \in I$, let a_i and b_i be two distinct points in X_i (possible since each X_i is not empty). Let a and b be two points in $\prod_{i \in I} X_i$ where the i^{th} component of a and b are a_i and b_i, respectively, and all other components of a and b are equal. Then $a \neq b$ and so, since $\prod_{i \in I} X_i$ is T_1 [T_0], there exists an open subset U of $\prod_{i \in I} X_i$ containing a but not b, and [or] an open subset V of $\prod_{i \in I} X_i$ containing b but not a. Furthermore there exist basis elements B_a and B_b (for the topology of $\prod_{i \in I} X_i$) such that $a \in B_a \subset U$ and $b \in B_b \subset V$. Therefore B_a contains a but not b, and [or] B_b contains b but not a. Now basis elements in $\prod_{i \in I} X_i$ are of the form $B_a = \prod_{k \in I} U_k$, $B_b = \prod_{k \in I} V_k$, where the U_k and V_k are open subsets of X_k (and $U_k = X_k$ and $V_k = X_k$ for all but finitely many values of $k \in I$ if $\prod_{i \in I} X_i$ is given the product topology, but that's not important in this proof). Since *a* and *b* only differ in their i^{th} component, then B_a and [or] B_b can only separate *a* and *b* if U_i contains *a* but not *b*, and [or] V_i contains *b* but not *a*. Thus X_i is T_1 [T_0]. Since $i \in I$ was arbitrary, then X_i is T_1 [T_0] for all $i \in I$.

$\Leftarrow$ Suppose X_j is T_1 [T_0] for all $j \in I$. Let $(x_j)_{j\in I}$ and $(y_j)_{j\in I}$ be two distinct points in $\prod_{i\in I} X_i$, where $\prod_{i\in I} X_i$ is given either the product topology or box topology. Then for some coordinate $i \in I$, we have $x_i \neq y_i$. But X_i is T_1 [T_0], and so there exist disjoint open sets *U* and [or] *V* of X_i such that *U* contains x_i but not y_i, and [or] *V* contains y_i but not x_i. Then $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are open subsets of $\prod_{i\in I} X_i$ (in either the product topology or box topology) such that $\pi_i^{-1}(U)$ contains *x* but not *y*, and [or] $\pi_i^{-1}(V)$ contains *y* but not *x*. Therefore $\prod_{i\in I} X_i$ is T_1 [T_0]. ■

12. (Milewski, p.501) Prove that every metric space is T_1 (and therefore T_0).

First solution (by me):

Every metric space is Hausdorff (proven in the Hausdorff Spaces chapter), and therefore T_1 and T_0. But we'll give a direct proof here. Let *x* and *y* be two distinct points of a metric space (X, d). Then $d(x, y) = r > 0$. Consequently, $B_d(x, r)$ contains *x* but not *y* and $B_d(y, r)$ contains *y* but not *x*. Thus *X* is a T_1-space. ■

Of course, using $r = \frac{1}{2}d(x, y)$ would prove that *X* is Hausdorff. It also turns out that *X* is also normal in addition to T_1, and hence is a T_4-space.

Second solution (Milewski):

In a metric space, the condition of a set *A* to be closed is that if $\lim_{n\to\infty} x_n = x$, where $x_n \in A$ for all $n \in \mathbb{Z}_+$, then $x \in \overline{A}$ (i.e. *A* contains all its limit points). Let $a \in A$. Since *a* is the only element of $\{a\}$, and clearly $\lim_{n\to\infty} a = a$, and $a \in \overline{\{a\}}$ by definition of closure, then $\{a\}$ is closed. Thus all single-point sets in *A* are closed and so *A* is a T_1-space. ■

13. (Munkres, p.99) Prove that if *A* is a subset of a T_1-space *X*, then *x* is a limit point of *A* if and only if every open subset of *X* containing *x* contains infinitely many points of *A*.

If every neighbourhood of *x* contains infinitely many points of *A*, then it contains at least two points of *A* and so contains at least one point of *A* other than *x* itself, and so *x* is a limit point of *A*.

Conversely, suppose that *x* is a limit point of *A*. Assume there exists an open subset *U* of *X* containing *x* that intersects *A* in only finitely many points. Then *U* also intersects $A - \{x\}$ at finitely many points, say at the points $x_1, ..., x_n$. Since *X* is a T_1-space, then $\{x_1, ..., x_n\}$ is closed by problem 6. Consequently,

$$U - \{x_1, ..., x_n\} = U \cap (X - \{x_1, ..., x_n\})$$

is the intersection of two open sets and hence is an open set (containing *x*). Since *U* intersects *A* only at $x_1, ..., x_n$ or at $x, x_1, ..., x_n$, then $U - \{x_1, ..., x_n\}$ is an open subset of *X* containing *x* that does not intersect *A* at any point except (possibly) at *x*. This contradicts the assumption that *x* is a limit point of *A*. Thus every open set containing *x* intersects *A* at infinitely many points. ■

14. (Milewski, p.503) Let X be a finite set. Prove that the only topology on X which makes X into a T_1-space is the discrete topology.

Let X be a finite T_1 space. Then every finite subset of X is closed by problem 6. Since X is a finite set, then every subset of X is closed, which means that every subset of X is open. Thus X has the discrete topology. ■

15. (Schaum's, p.143) Show that a finite subset of a T_1-space X has no accumulation points.

First solution (by me):

Let $A = \{a_1, ..., a_n\}$ be a finite subset of a T_1-space X. We show that every point in A is not a limit point. Let $a_i \in A$. Since X is a T_1-space, then for all $j \neq i$, there exists an open subset U_j of X containing a_i such that $a_j \notin U_j$. Let $U = \bigcap_{j \neq i} U_j$, which is an open set since it is the intersection of $n-1$ open sets. Now U does contains a_i but U does not contain any other point in A, since if $a_k \in U$ for some $k \neq i$, then $a_k \in \bigcap_{j \neq i} U_j \subset U_k$, a contradiction. Thus a_i is not a limit point. Thus A contains no limit point. ■

Second solution (by Schaum's):

Let $A = \{a_1, ..., a_n\}$ be a finite subset of a T_1-space X. The subset $\{a_1, ..., a_{i-1}, a_{i+1}, a_n\}$ is a finite subset of X for any i, and thus is a closed subset of X by problem 6. Consequently, $X - \{a_1, ..., a_{i-1}, a_{i+1}, a_n\}$ is an open subset of X, and since it contains a_i and no other point of A, then a_i is not a limit point of A. Thus A has no limit point. ■

Third solution (by me):

By problem 0, then x is a limit point of X if and only if every open set containing x contains infinitely many points of X, which is not possible since A has only finitely many points. Thus A has no limit point. ■

16. (Schaum's, p.144) Let X be a T_1-space and $\mathcal{B}_x$ a local basis at $x \in X$. Prove that for every $y \in X$ distinct from x, there is some member of $\mathcal{B}_x$ that does not contain x.

Since $x \neq y$, and X be a T_1-space, then there exists an open set U containing x but not y. Consequently, there is a local basis element $B \in \mathcal{B}_x$ such that $x \in B \subset U$ and so B does not contain y. ■

17. (Schaum's, p.144) Let X be a first countable T_1-space and let A be a subset of X. Show that if $p \in X$ is a limit point of A, then there exists a sequence of distinct terms in A that converges to p.

Since X is first countable, there exists a nested local basis $\mathcal{B} = \{B_n \mid n \in \mathbb{Z}_+\}$ at $x \in X$. Set $B_{i_1} = B_1$. Since x is a limit point of A, then B_{i_1} contains a point $a_1 \in A$ different from x. Since X is a T_1-space, then by problem 16, there exists $B_{i_2} \in \mathcal{B}$ such that $a_1 \notin B_{i_2}$. Similarly B_{i_2} contains a point $a_2 \in A$

different from *x* and, since $a_1 \notin B_{i_2}$, different from a_1. Again, by problem 16, there exists $B_{i_3} \in \mathscr{B}$ such that $a_2 \notin B_{i_3}$. Furthermore, $a_2 \in B_{i_2}$ and $a_2 \notin B_{i_3}$ means that $B_{i_3} \subset B_{i_2}$.

Continuing in this manner we obtain a sequence $\{B_{i_1}, B_{i_2}, ...\}$ of $\mathscr{B}$ and a sequence $\{a_1, a_2, ...\}$ of distinct terms in *A* with $a_n \in B_{i_n}$ and $B_{i_{n+1}} \subset B_{i_n}$ for all $n \in \mathbb{Z}_+$. Then for any open set *U* containing *x*, there exists $B_n \in \mathscr{B}$ such that $x \in B_n \subset U$, and consequently some B_{i_m} such that $x \in B_{i_m} \subset B_n \subset U$. Therefore we have $a_k \in B_{i_m} \subset U$ for all $k \geq m$. Thus the sequence $\{a_1, a_2, ...\}$ converges to *x*. ■

18. (Adamson, p.45) Let *X* be a T_1-space. Let *A* be a subset of *X* and let $x \in \overline{A}$. Prove that if $x \notin A$, then every open subset of *X* containing *x* contains infinitely many points of *A*.

First solution (by Adamson):

Suppose there exists an open subset *U* containing *x* that intersects *A* at finitely many points, say at $a_1, ..., a_n$. Since $x \notin A$, then none of these points are *x*. Since *X* is T_1, then by problem 16, there exist basis elements $B_1, ..., B_n$ for the topology of *X* containing *x* such that $a_i \notin B_i$ for all $i = 1, ..., n$. Then $(\bigcap_{i=1}^{n} B_i) \bigcap U$ is an open subset of *X* (since *n* is finite) containing *x* that does not contain any of the points $a_1, ..., a_n$, since $a_k \in (\bigcap_{i=1}^{n} B_i) \bigcap U$ for some *k* means that $a_k \in \bigcap_{i=1}^{n} B_i \subset B_k$, a contradiction. But $(\bigcap_{i=1}^{n} B_i) \bigcap U \subset U$, and *U* only intersects *A* at the points $a_1, ..., a_n$. Thus $(\bigcap_{i=1}^{n} B_i) \bigcap U$ does not intersect *A* at all. But since $x \in \overline{A}$, every open subset of *X* containing *x* intersects *A*. This contradiction means that every open subset of *X* containing *x* must contain infinitely many points of *A*. ■

Second solution:

Suppose there exists an open subset *U* containing *x* that intersects *A* at finitely many points, say at $a_1, ..., a_n$. Since $x \notin A$, then none of these points are *x*. Since *X* is a T_1-space, then $\{a_1, ..., a_n\}$ is closed by problem 6. Consequently,

$$U - \{a_1, ..., a_n\} = U \bigcap (X - \{a_1, ..., a_n\})$$

is the intersection of two open sets and hence is an open set (containing *x*). Since *U* intersects *A* only at $a_1, ..., a_n$, then $U - \{a_1, ..., a_n\}$ is an open subset of *X* containing *x* that does not intersect *A* at all, contradicting the fact that $x \in \overline{A}$. Thus every open set containing *x* intersects *A* at infinitely many points. ■

19. (Adamson, p.44) Prove (X, d) be a pseudometric space, where a pseudometric is a function that satisfies the axioms of a metric except two distinct points may have zero distance. Prove that *X* is a T_0-space if and only if *d* is a metric.

$\Rightarrow$ Suppose (X, d) is a T_0-space. Let *x* and *y* be two distinct points in *X*. Since *X* is a T_0-space, there exists an open set *U* that either contains *x* but not *y*, or contains *y* but not *x*. So assume $x \in U$ but $y \notin U$. Then there exists $\varepsilon > 0$ such that $x \in B_d(x, \varepsilon) \subset U$. Thus $y \notin B_d(x, \varepsilon)$ so that $d(x, y) > \varepsilon > 0$. Thus *d*, together with the pseudometric properties, is a metric.

Adamson: If *d* is not a metric, then there exist two distinct points *x* and *y* such that $d(x, y) = 0$. Then for any $\varepsilon > 0$, we have $x \in B_d(y, \varepsilon)$ and $y \in B_d(x, \varepsilon)$. Thus any basis element for the topology of *X* that contains one point also contains the other. Thus X cannot be a T_0-space. This contradiction means that d must be a metric.

$\Leftarrow$ Suppose that d is a metric. Then (X,d) is a metric space and hence is a T_0-space by problem 12. ■

20. (Adamson, p.45) Prove that a *quasimetric space* is a T_1-space, where a quasimetric is a function that satisfies the axioms of a metric except the symmetry property of a metric.

Let (X,d) be a quasimetric space. Let x and y be two distinct points in X. Then $r = d(x,y) > 0$ and $s = d(y,x) > 0$. Thus $B_d(x,r) = \{z \in X \mid d(x,z) < r)$ is an open set containing x but not y, and $B_d(y,q) = \{z \in X \mid d(y,z) < s)$ is an open set containing y but not x. Thus X is a T_1-space. ■

Note: Not all quasimetric spaces are Hausdorff, and Adamson provides a counterexample in problem 144.

21. (Adamson, p.45). Let X be a T_1-space with a finite basis for its topology. Prove that X is finite and has the discrete topology.

Let $\mathscr{B} = \{B_1,\ldots,B_n\}$ be a finite basis for X. Let $x \in X$. Then there exists $B_{i_1} \in \mathscr{B}$ such that $x \in B_{i_1}$. If $B_{i_1} \neq \{x\}$, then choose a point $x_1 \neq x$ such that $x_1 \in B_{i_1}$. Since X is T_1, then there exists an open set U containing x but not x_1. Since $U \cap B_{i_1}$ is open subset of X containing x, then there exists $B_{i_2} \in \mathscr{B}$, such that $x \in B_{i_2} \subset (B_{i_1} \cap U)$. Then $x \in B_{i_2} \subset U$, so $x_1 \notin B_{i_2}$. If $B_{i_1} \neq \{x\}$, then we may repeat this process, so obtaining a strictly decreasing sequence of basic open sets.

Since $\mathscr{B}$ is finite, then this process must stop with $x \in B_{i_k} \underset{\neq}{\subset} B_{i_{k-1}} \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} B_{i_1}$. If $B_{i_k} \neq \{x\}$, then we can choose $x_k \in B_{i_k}$ distinct from x and then there must exist an open set V containing x but not x_k, and so (by definition of $\mathscr{B}$ being a basis) there must exist $B_{i_{k+1}} \in \mathscr{B}$ such that $x \in B_{i_{k+1}} \subset (B_{i_k} \cap V)$, and thus $x \in B_{i_{k+1}} \underset{\neq}{\subset} B_{i_k} \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} B_{i_1}$ and the process is continuing, a contradiction. Thus we must have $B_{i_k} = \{x\}$. Thus $\{x\}$ is an open subset of X. Since x was an arbitrary point in X, then X has the discrete topology. Furthermore, since $\mathscr{B}$ is finite, and $\{\{x\} \mid x \in X\} \subset \mathscr{B}$, then X must be finite. ■

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Minor Theorems:

24. (Adamson, p.43) Prove that if X has any particular point topology, then X is a T_0-space, where the particular point topology corresponding to $x_0 \in X$ is defined by

$$\mathscr{T}_{x_0} = \{U \in \mathscr{P}(X) \mid x_0 \in U\} \cup \{\varnothing\}.$$

Furthermore, show that X is not a T_1-space (if X has more than one point).

If X does not have more than one point, then the T_0 property of X is vacuously true. Otherwise, let $x, y \in X$ be two distinct points in X. If either of these two points is x_0, say $x = x_0$, then $\{x_0\}$ is an open set containing x but not y. If none of x and y are x_0, then $\{x_0, x\}$ is an open set containing x but not y (and $\{x_0, y\}$ is an open set containing y but not x, though this is not needed). Thus X is a T_0-space.

Now if X has more than one point, then for any $x \notin x_0$, any open set containing x must contain x_0, and thus X is not a T_1-space. ■

25. (Adamson, p.43-44) Given a space X, define $\sim$ by $x \sim y$ iff $\overline{\{x\}} = \overline{\{y\}}$.

a) Prove that $\sim$ is an equivalence relation.

Reflexivity: Let $x \in X$. Then $\overline{\{x\}} = \overline{\{x\}}$ so $x \sim x$.

Symmetry: Suppose $x \sim y$. Then $\overline{\{x\}} = \overline{\{y\}}$, so $\overline{\{y\}} = \overline{\{x\}}$. Thus $y \sim x$.

Transitivity: Suppose $x \sim y$ and $y \sim z$. Then $\overline{\{x\}} = \overline{\{y\}}$ and $\overline{\{y\}} = \overline{\{z\}}$. Thus $\overline{\{x\}} = \overline{\{y\}} = \overline{\{z\}}$ so $x \sim z$. ■

b) Prove that the canonical surjection $q : X \to X/\sim$ is an open map.

Let U be an open subset of X. We claim that $q^{-1}(q(U)) = U$, from which the result follows since q is a quotient map. Now $U \subset q^{-1}(q(U))$ already holds from elementary set theory (with equality holding if q is injective—but since q is not injective, we must show the reverse inclusion directly). Let $x \in q^{-1}(q(U))$. Then $q(x) \in q(U)$ so that $q(x) = q(y)$ for some $y \in U$. Then $x \sim y$ so $\overline{\{x\}} = \overline{\{y\}}$ by definition of $\sim$. Thus $y \in \overline{\{y\}} = \overline{\{x\}}$, so that every open subset of X containing y intersects $\{x\}$, i.e. contains x. In particular, U contains y and thus also contains x. Thus $x \in U$ so that $q^{-1}(q(U)) \subset U$. Therefore $q^{-1}(q(U)) = U$. Since U is a quotient map, then $q(U)$ must be an open subset of Y. Thus q is an open map. ■

c) Prove that the resulting quotient space $X/\sim$ is T_0.

Let $[x] = q(x)$ and $[y] = q(y)$ be two distinct points in $X/\sim$. Then $x \nsim y$ and thus $\overline{\{x\}} \neq \overline{\{y\}}$. By problem 5, there exists an open subset of X that contains x and not y, or contains y or not x. So assume U is an open subset of X such that $x \in U$ and $y \notin U$. By part (b), $q(U)$ is an open subset of $X/\sim$ containing $q(x) = [x]$. We shall show that $q(U)$ does not contain $[y]$. Suppose $[y] \in q(U)$. Then $q(y) = [y] = q(z)$, for some $z \in U$. Then $y \sim z$ so $\overline{\{y\}} = \overline{\{z\}}$. Thus $z \in \overline{\{z\}} = \overline{\{y\}}$, so that every open subset of X containing z intersects $\{y\}$, i.e. contains y. But $z \in U$ and $y \notin U$, a contradiction. Thus $q(U)$ is an open subset of $X/\sim$ that contains $[x]$ but does not contain $[y]$. Thus $X/\sim$ is a T_0-space. ■

26. (Conover, 162) Prove that a subset A of a T_1-space is countably compact if and only if every countable open cover of A has a finite subcover.

Proof by Morphism:

$\Rightarrow$ Let A be a countably compact subset of a T_1-space X. If A is finite, then clearly every countable open cover of A has a finite subcover by taking one member of the open cover to cover each point of A. So assume A is infinite. Let $\mathcal{U} = \{U_1, U_2, ...\}$ be a countable cover of A. Suppose no finite subcollection of $\mathcal{U}$

covers *A*. Then there exists a point a_1 in *A* that is not covered by $\{U_1\}$ (since $\{U_1\}$ cannot cover *A*), a point a_2 (different from a_1) in *A* that is not covered by $\{U_1,U_2\}$ (since $\{U_1,U_2\}$ cannot cover *A*), …, and in general a point a_n (different from $a_1,...,a_{n-1}$) in *A* not covered by $\{U_1,...,U_n\}$ for all $n\in\mathbb{Z}_+$ (since $\{U_1,...,U_n\}$ cannot cover *A*). Then $\{a_n\}=\{a_1,a_2,...\}$ is an infinite subset of *A* and thus, since *A* is countably compact, must have a limit point *a* in *A*. Since $\boldsymbol{\mathcal{U}}$ covers *A*, then $a\in U_k$ for some $k\in\mathbb{Z}_+$. Now by construction of $\{a_n\}$, for all $i>k$, a_i is not covered by $\{U_1,...,U_i\}$ and thus $a_i\notin U_k$ for all $i>k$. Thus U_k can only intersect $\{a_n\}$ at the points $a_1,...,a_k$. But *X* is a T_1-space, so by problem 13, U_k must intersect $\{a_n\}$ at infinitely many points since *a* is a limit point of $\{a_n\}$ and U_k is an open subset of *X* containing *a*. This contradiction means that there must be a finite subcollection of $\boldsymbol{\mathcal{U}}$ that covers *A*.

Second proof of this direction:

Let *A* be a countably compact subset of a T_1-space *X*. If *A* is finite, then clearly every countable open cover of *A* has a finite subcover by taking one member of the open cover to cover each point of *A*. So assume *A* is infinite. Let $\boldsymbol{\mathcal{U}}=\{U_1,U_2,...\}$ be a countable cover of *A*. Given any finite subcollection $\boldsymbol{\mathcal{V}}$ of $\boldsymbol{\mathcal{U}}$, let *S* be the set of all points in *A* that is not covered by $\boldsymbol{\mathcal{V}}$. If $S=\varnothing$, then $\boldsymbol{\mathcal{V}}$ is a finite subcover of *A* and we are done. If *S* is finite, then each point in *S* is contained in some member of $\boldsymbol{\mathcal{U}}$ and so there is a finite subcollection $\boldsymbol{\mathcal{W}}$ of $\boldsymbol{\mathcal{U}}$ covering *S* and so $\boldsymbol{\mathcal{V}}\bigcup\boldsymbol{\mathcal{W}}$ is a finite subcover of *A*. Thus assume *S* is infinite.

Since *A* is countably compact and *S* is infinite, then *S* has a limit point a_1 in *A*, which must belong to some $U_{i_1}\in\boldsymbol{\mathcal{U}}$. Since U_{i_1} is an open subset of *X* containing a_1 and a_1 is a limit point of *S*, then U_{i_1} must intersect *S*. If $S-U_{i_1}$ is finite, then again we conclude that $\boldsymbol{\mathcal{U}}$ admits a subcover of *A*. If $S-U_{i_1}$ is infinite, then again by countably compactness of *A*, $S-U_{i_1}$ has a limit point a_2 in *A*. Now a_2 cannot be in U_{i_1} since then U_{i_1} would be an open subset of *X* containing a_2 which does not intersect $S-U_{i_1}$, contradicting the fact that a_2 is a limit point of $S-U_{i_1}$. Thus a_2 must belong to U_{i_2} for some $U_{i_2}\in\boldsymbol{\mathcal{U}}$ distinct from U_{i_1}. If $S-U_{i_1}\bigcup U_{i_2}$ is finite, then once again we conclude that $\boldsymbol{\mathcal{U}}$ admits a subcover of *A*. If instead $S-U_{i_1}\bigcup U_{i_2}$ is infinite, then again by countably compactness of *A*, $S-U_{i_1}\bigcup U_{i_2}$ has a limit point a_3 in *A*. Now a_3 cannot be in U_{i_1} or U_{i_2} since then $U_{i_1}\bigcup U_{i_2}$ would be an open subset of *X* containing a_2 which does not intersect $S-U_{i_1}\bigcup U_{i_2}$, contradicting the fact that a_3 is a limit point of $S-U_{i_1}\bigcup U_{i_2}$. Thus a_3 must belong to U_{i_3} for some $U_{i_3}\in\boldsymbol{\mathcal{U}}$ distinct of U_{i_1} and U_{i_2}. We continue this process to obtain the (possibly terminating) sequence of points $\{a_n\}=\{a_1,a_2,a_3,a_4,...\}$.

Suppose $\{a_n\}$ is infinite, which means that $a_n\in U_{i_n}$ for all $n\in\mathbb{Z}_+$. Now since $a_i\in A$ for all $i\in\mathbb{Z}_+$, then $\{a_n\}$ is an infinite subset of *A*, and so by countably compactness of *A*, $\{a_n\}$ must have a limit point *a* in *A*. Now *a* must be in U_{i_k} for some $U_{i_k}\in\boldsymbol{\mathcal{U}}$. Since every a_n cannot belong to $U_1,...,U_{n-1}$ by the previous paragraph, then U_{i_k} cannot contain a_n for all $n>i_k$. Thus U_{i_k} can only intersect $\{a_n\}$ at the points $a_1,...,a_{i_k}$. But *X* is a T_1-space, so by problem 13, U_{i_k} must intersect $\{a_n\}$ at infinitely many points since *a* is a limit point of $\{a_n\}$ and U_{i_k} is an open subset of *X* containing *a*. This contradiction means that $\{a_n\}$ cannot be infinite. Thus $\{a_n\}$ must terminate at some last term a_N. This means that there is no a_{N+1} to be a limit point of $S-\bigcup_{j=1}^{N}U_{i_j}$, which means that $S-\bigcup_{j=1}^{N}U_{i_j}$ must be finite (because if $S-\bigcup_{j=1}^{N}U_{i_j}$ were infinite, then it would have a limit point a_{N+1} by countably compactness of *A*). Thus there is a finite subcollection $\boldsymbol{\mathcal{Y}}$ of $\boldsymbol{\mathcal{U}}$ that covers $S-U_{i_N}$, and so $\{U_{i_1},...,U_{i_N}\}\bigcup\boldsymbol{\mathcal{Y}}$ is a finite subcollection of $\boldsymbol{\mathcal{U}}$ that covers *A*.

$\Leftarrow$ Suppose X is a T_1-space and that every open cover of a subset A has a finite subcover. Let B be an infinite subset of A. Suppose B has no limit point in A. Then for any $b \in B$, we can find an open subset U_b of X that intersects B only at $\{b\}$. Using the Axiom of Choice, let $C = \{b_1, b_2, ...\}$ be any countable subset of B. Now B contains all its limit points (since it has none) and thus is a closed subset of X. Since X is a T_1-space, then $\{b_n\}$ is a closed subset of X for all $n \in \mathbb{Z}_+$. Thus

$$C = \{b_1, b_2, ...\} = B \bigcap_{n \in \mathbb{Z}_+} \{b_n\}$$

is the intersection of closed subsets of X and hence is a closed subset of X. Thus

$$\mathcal{U} = \{U_{b_1}, U_{b_2}, ...\} \bigcup \{A - C\}$$

is a collection of open sets. Furthermore $\mathcal{U}$ is a countable collection, and, since $b_n \in U_{b_n}$ for all $n \in \mathbb{Z}_+$, $\mathcal{U}$ is a countable open cover of A. However any finite subcollection of $\mathcal{U}$ will have to have some U_{b_n} deleted, and deleting any U_{b_n} from $\mathcal{U}$ will leave the point $b_n \in A$ uncovered. Thus $\mathcal{U}$ is a countable open cover of A that has no finite subcover, contrary to hypothesis. Thus B must have a limit point in A. Therefore every infinite subset of A has a limit point in A, and so A is countably compact. ■

27. (Adamson, p.44) Let X be a topological space and define a relation R on X by setting $(x, y) \in R$ if and only if $x \in \overline{\{y\}}$.

a) Prove that R is reflexive and transitive.

First we prove that R is reflexive and transitive. By definition of closure, $x \in \overline{\{x\}}$ for all $x \in X$ so so $(x, x) \in R$, so that R is reflexive. Now suppose $(x, y) \in R$ and $(y, z) \in R$. Then $x \in \overline{\{y\}}$ and $y \in \overline{\{z\}}$. Now $x \in \overline{\{y\}}$ means that any open set U containing x intersects $\{y\}$, i.e. contains y. But $y \in \overline{\{z\}}$ means that, since U contains y, U intersects $\{z\}$. Thus every open set containing x also intersects $\{z\}$, which means that $x \in \overline{\{z\}}$. Thus $(x, z) \in R$, and so R is transitive. Another way to show that $x \in \overline{\{z\}}$, is to note that since $\{y\} \subset \overline{\{z\}}$, then $\overline{\{y\}} \subset \overline{\overline{\{z\}}} = \overline{\{z\}}$, and so $x \in \overline{\{y\}} \subset \overline{\{z\}}$. ■

b) Prove that R is a partial order relation if and only if X is a T_0-space.

$\Rightarrow$ Suppose R is a partial order relation. Let x and y be two distinct points in X. Then by anti-symmetry of R, we cannot have $(x, y) \in R$ and $(y, x) \in R$, i.e. we cannot have both $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Thus we must have either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Then by problem 4, X is a T_0-space.

$\Leftarrow$ Suppose X is a T_0-space. By part (a), R is reflexive and transitive, so we need only show that R is antisymmetric to prove that R is a partial order relation. Suppose $(x, y) \in R$ and $(y, x) \in R$. Then $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. If $x \neq y$, then there exists an open set U that either contains x but not y, or contains y but not x. But $x \in \overline{\{y\}}$ means that every open set containing x intersects $\{y\}$, i.e. contains y and $y \in \overline{\{x\}}$ means that every open set containing y intersects $\{x\}$, i.e. contains x. This contradiction means that we must have $x = y$ and so R is antisymmetric and thus is a partial order relation.

Adamson: $x \in \overline{\{y\}}$ means that $\{x\} \subset \overline{\{y\}}$, so that $\overline{\{x\}} \subset \overline{\overline{\{y\}}} = \overline{\{y\}}$. Similarly, $y \in \overline{\{x\}}$ means that $\{y\} \subset \overline{\{x\}}$, so that $\overline{\{y\}} \subset \overline{\overline{\{x\}}} = \overline{\{x\}}$. Thus $\overline{\{x\}} = \overline{\{y\}}$. Since X is a T_0-space, then by problem 5, distinct one-point subsets of X have distinct closures. Thus if $x \neq y$, then $\{x\} \neq \{y\}$ so that $\overline{\{x\}} \neq \overline{\{y\}}$. Hence we must have $x = y$, and so R is antisymmetric and thus is a partial order relation. ■

c) A topological space X is called an *Alexandrov space* if the intersection of every collection of open subsets of X is open. Prove that a subset C of an Alexandrov space is closed if and only if whenever $y \in C$ and $(x, y) \in R$, then $x \in C$.

$\Rightarrow$ Suppose C is a closed subset of an Alexandrov space X. Let $y \in C$ and $(x, y) \in R$, i.e. $x \in \overline{\{y\}}$. Since $y \in C$ and C is closed, then $\overline{\{y\}} \subset C$ by definition of closure. Thus $x \in \overline{\{y\}} \subset C$.

$\Leftarrow$ Let C be a subset of an Alexandrov space X, and suppose that $y \in C$ and $(x, y) \in R$, i.e. $x \in \overline{\{y\}}$, implies $x \in C$. We need to show that C is closed. Now for any collection $\{C_i \mid i \in I\}$ of closed subsets of X, we have by deMorgan's Law,

$$\bigcup_{i \in I}(C_i) = X - \bigcap_{i \in I}(X - C_i),$$

which is the complement of an intersection of open sets (and an intersection of open sets is open since X is an Alexandrov space) and hence is closed. Consequently, $\bigcup_{y \in C} \overline{\{y\}}$ is a union of closed subsets of X and hence is a closed set containing C. Thus $\overline{C} \subset \bigcup_{y \in C} \overline{\{y\}}$. Then for any $x \in \overline{C}$, we have $x \in \overline{\{y\}}$ for some $y \in C$. Thus by hypothesis, we have $x \in C$. Hence $\overline{C} \subset C$. Since $C \subset \overline{C}$, then we have $C = \overline{C}$. Therefore C is closed. ■

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