The Deeper Implications of Gödel's Theorems

By Kenton Hirowatari

Math 480

Fall 2007

Kurt Gödel is said to be the "greatest mathematical logician of all time" (Hawking 1089) and is best know for his Incompleteness Theorems. Gödel's theorems have implications on the foundations of mathematics. We will examine the time period of the Gödel's theorems in view of formalism, understand what two of his theorems are saying, and analyze where exactly his incompleteness theorems are applicable and also where they are not applicable.

It is important to note that Gödel's Incompleteness theorems were published in 1931 at a time when many mathematicians were trying to rigorize mathematics. One notable person of the era was David Hilbert. At an international congress in Paris, Hilbert gave a talk entitled *Mathematical Problems* which contained 23 problems, which he used to motivate a formalist foundation of mathematics, also known as Hilbert's Programme.

> "These rules form a closed system that can be discovered and definitively stated. The fundamental idea of my [Programme] is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds… If any totality of observations and phenomena deserves to

> be made the object of a serious and thorough investigation, it is this one…" (Shapiro, 299).

With respect to formalism, one problem of great importance was Hilbert's 2nd, which asked mathematicians and logicians:

> "1. To prove that all true mathematical statements could be proven, that is the completeness of mathematics
>
> 2. To prove that only true mathematical statements could be proven, that is, the consistency of mathematics" (Hawking 1121).

This accomplishment would lead to a complete and consistent system: all paradoxes would be avoided and more importantly no paradoxes could possibly arise.

To understand what Hilbert meant by a "complete and consistent system" we must understand exactly what consistency and completeness is. Consistency can be defined to be "when every theorem, upon interpretation, comes out true" (Hofstadter, 101). More specifically, if a system is complete, then every statement which can be derived from this system is either true of false and can be shown to be either true or false using the axioms of the system. Next we must define what it means for a system to be complete. Completeness with respect to axiomatic systems is best defined as "when all statements which are true… and which can be expressed as well-formed [and meaningful] strings of the system, are theorems" (Hofstadter, 101). It follows that, a system is inconsistent if a false proposition can be derived from the axioms of that system. Hilbert's Programme was proposed in 1900 and for the next 30 years many were contributing to the proof of this result.

Bertrand Russell and Alfred North Whitehead gave a noble and elaborate attempt in *Principia Mathematica* (PM) to show that the 2nd problem of Hilbert's Programme was indeed true. In PM the aim was to derive the fundamentals of arithmetic (of natural numbers) from logical principles. This was done with an extreme amount of rigor but ultimately did not explicitly show that no contradictions could arise, thus leaving Hilbert's 2nd question unanswered.

Also notable in that era were Zermelo and Frankel who developed a set theory called Zermelo-Frankel set theory (ZF). This set theory was based on only 8 axioms and rid set theory of certain paradoxes such as "the set of all sets". By the 19th century most of mathematics was rigorized enough to avoid paradoxes which were rampant earlier in the century and it was thought by some that this could be continued until all of math was complexly free of paradoxes and thus be reduced to a relatively small number of axioms. Thus Hilbert Programme seemed to be in motion and possibly close to fruition.

Gödel himself admitted in his famous paper *On the Formally Undecidable Propositions of Principia Mathematica and Related Systems* that: "one might conjecture that these axioms [PM and ZF] and rules of inference are sufficient to decide any mathematical question that can at all be formally expressed in this system" (van Heijenoort, 597). However we now arrive at the death blow to Hilbert's Programme. In 1931 Gödel published his famous incompleteness Theorem stating "For any consistent formal, computably enumerable theory that proves basic arithmetical truths [of natural numbers], an arithmetical statement that is true, but not provable in the theory, can be constructed. That is, any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete". In other words, for any non-trivial mathematical system which can describe the arithmetic of natural numbers, the system cannot be both consistent and complete.

We can take one application of this in relation to Peano Arithmetic (PA). PA is based on 5 axioms and from these we can derive addition, subtraction, multiplication, distribution, commutativity, associativity, etc. Gödel showed that we could construct a statement in PA stating 'this statement is unprovable'. Let's call this statement 'P'. Then if P is true, thus unprovable, then the PA is incomplete: that is, there is a meaningful statement which can be made within PA which cannot be shown to be true or false. On the other hand, if P is false, then we arrive at a version of the liar's paradox; thus, PA would be inconsistent. Therefore, Gödel's incompleteness theorem implies that PA must be either incomplete or inconsistent.

By now we may have been tempted to ask why we wouldn't simply add an axiom to PA that says "P is true"; let's call this statement P*. This leads us to Gödel's 2nd Incompleteness theorem: "If formal arithmetic is consistent then that consistency cannot be proven from within formal arithmetic" (Hawking 1091). Thus, if we can 'prove' that a system is consistent from formal arithmetic, we have shown it to be inconsistent by Gödel's 2nd Incompleteness theorem. Hence, in our example of adding P* to PA we have then contradicted Gödel's 2nd Incompleteness theorem and shown our modified PA to also be incomplete.*

One other note we must make about Gödel's Incompleteness theorem is that:

> "this situation is not in any way due to the special nature of the systems that have been set up but hold for a wide class of formal systems; among these, in particular, are all system that result from the two mentioned through the addition of a finite number of axioms" (van Heijenoort, 597).

That is to say that Gödel's theorem did not find one isolated flaw in ZF and PM but simply used these as an example. Also, his results could be extended to other systems. In fact, Gödel's theorem applies to any formal system that can be expressed in a finite set of axioms as strong as PA and expressing integer arithmetic (Rucker, 268).

Thus we see Hilbert's Programme effectively destroyed*[ref]. No system which can make meaningful arithmetic statements of natural numbers could be both complete and consistent. It's worth noting that there are some small systems that can describe arithmetic of natural numbers however this can only be done in a much more limited way.

We can now see the consequence of his theorem: math cannot and will not ever be rigorized to a finite set of axioms! We will never rid math completely of all possible paradoxes. One implication of Gödel's theorem is taken directly by him and relates to the Continuum Hypothesis. The Continuum Hypothesis states that there is no cardinality (amount of elements) between the cardinality of the natural numbers ($\aleph_0$) and the cardinality of the real numbers (c). For example, Gödel showed that the Continuum hypothesis could be added to the ZFC (Zermelo-Frankel with the Axiom of Choice) and this would not lead to any contradiction (Hawking 1093). It was also shown later that removing the Continuum hypothesis from ZFC would not yield any contractions. This shows ZFC to be incomplete. That is, there is a statement that can be made within the system that cannot be proven to be true or false. We call these undecidable statements.

Another statement shown to be undecidable by Gödel is the Axiom of Choice; which states that we may choose an element from a set. Gödel proved in 1940 that the Axiom of Choice is "consistent with other axioms of set theory" (Boyer, 611). This is especially important in the area of analysis where the Axiom of Choice is indispensable.

We have seen some remarkable, while straightforward results of Gödel's Incompleteness theorem; however, this theorem is often a misunderstood and misapplied theorem. There are many misconceptions of what his theory applies to, but now that we know the definite meaning of his theorem we can analyze these misconceptions. One common conjecture that is made upon being introduced to Gödel's theorems is that we cannot prove anything in mathematics or that we cannot be certain that our results are correct. To take this further is to say if mathematics cannot be shown to be consistent, then it must all be completely inconsistent, full of paradoxes and flawed to the core. This is not at all what Gödel proved. Indeed, Gödel showed that logically we cannot create a system in mathematics, expressing integer arithmetic, which is both complete and consistent; that is, if a system is complete it is inconsistent and if it is consistent then it is incomplete. We can simply accept that in using ZFC or PA or any other consistent set of axioms, we will have an incomplete system. This is not to say that the axioms will infer falsities, only that there will always be truths which cannot be shown to be true.

A second misconception is that Gödel proved that the bible is inconsistent and thus leads to contradictions. This is a very silly statement because the bible is trivially incomplete with respect to formal systems. That is to say propositions could be made that the bible does not claim. For example the proposition "Joseph sneezed on his 19th birthday". The proposition is either true or false, of course, but the bible makes no claim about it and the claim cannot be derived from the bible. It is clearly not a system and Gödel says nothing in his incompleteness theorem about the bible. Similar misapplications of Gödel's theorem have also been applied to the Canadian constitution, Capitalism and many other trivially incomplete systems.

Another very common blunder that is made upon encountering Gödel's incompleteness theorem is the confusion between truth and provability. K. C. Cole in his book *The Universe and the Teacup: the Mathematic of Beauty and Truth* say "[t]his confidence that truths would be discovered in all fields was shattered by the recognition that there's no truth in mathematics" (Cole, 162). Cole erroneously equates truth and provability! Gödel did not show that mathematics has no truth in it, but, only that there may exist truths in a specific system can be shown to be true.

Indeed, two apples taken with another two apples are still four and can be trivially abstracted into mathematic as 2+2=4. That is both a true and mathematical statement. Indeed, we now notice one of the intricacies of Gödel's theorem is that "by focusing on provability rather than on truth, Gödel's sentence avoids the [liars'] paradox" (Hawking 1092). We can see that applications such as these are vague extrapolations of Gödel's theorems which are likely used for the author's intent instead of objective applications.

Other misinterpretation of Gödel's theorem is that Gödel has shown that no axiomatic system can be complete and consistent. On the contrary, Alfred Tarski showed that (first-order) arithmetic of real numbers (called the theory of real closed fields), for example, is consistent and complete. This does not contradict Gödel's Incompleteness Theorems because real number arithmetic doesn't allow you to formalize the notion of "integer", and thus it cannot fully express integer arithmetic. Tarski also gives an axiomization of Euclidean geometry which is equivalent to the theory of real closed fields, and so, is also consistent and complete. Thus, there are non-trivial, complete and consistent systems; however, these systems fall outside of the implications of Gödel's Incompleteness theorems.

Thus we have seen that Gödel has shown PA and ZFC to be incomplete. As well we have seen how the Continuum Hypothesis and the axiom of choice are independent of ZFC. Also, Gödel's incompleteness should not be extrapolated to situations which do not involve axiomatic systems or systems which do not express integer arithmetic, among other requirements. We have seen that although Gödel had a wide spread effect on mathematics, meta-mathematics, and mathematical logicism, his implications should not be taken too far. Lastly, a consequence of Gödel's theorem is that we as mathematicians will always have more to research. There will always be more to know about mathematics. We will always have surprising results that come out of mathematics. We will always be faced with challenges in mathematics where no one has gone before. There will always be 'ah ha' moments when we make a connection that no one has made before. In short, an implication of Gödel's theorem means we will never reach the end of mathematics.

Word Count: 2282

Works Cited

Boyer, Carl. *A History of Mathematics*. 1968 John Wiley & Sons Inc.:New York, 1986.

Cole, K. C.. *The Universe and the Teacup.* 1998 Harcourt Brace & Company: New York, 1998

Hawking, Stephen. *God Created the Integers*. 2005 Pennsylvania: Running Press, 2005.

Hofstadter, Douglas. *Gödel, Escher, Bach.* 1979 New York: Basic Books, 1979

Rucker, Rudy. *Infinity and the Mind.* 1986 New Jersey: Princeton University Press, 1995.

Stewart, Sharpio. *The Oxford Handbook of Philosophy of Mathematics and Logic.* 2005 Oxford: Oxford University Press. 2005.

van Heijenoort, Jean. *From Frege to Gödel.* 1967 Massachusette: Harvard University Press, 1967.