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The deformation of Poincaré subgroups concerning very special relativity

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We investigate here various kinds of semi-product subgroups of Poincaré group in the scheme of Cohen-Glashow's very special relativity along the deformation approach by Gibbons- Gomis-Pope. For each proper Poincaré subgroup which is a semi-product of proper lorentz group with the spacetime translation group $T(4)$, we investigate all possible deformations and obtain all the possible natural representations inherited from the $5-d$ representation of Poincaré group. We find from the obtained natural representation that rotation operation may have additional accompanied scale transformation when the original Lorentz subgroup is deformed and the boost operation gets the additional accompanied scale transformation in all the deformation cases. The additional accompanied scale transformation has a strong constrain on the possible invariant metric function of the corresponding geometry and the field theories in the spacetime with the corresponding geometry.

very special relativity, deformation of Poincare subgroups, natural representation, accompanied scale transformation

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1 Introduction

The local Lorentz symmetry and CPT invariance is one of the fundamentals of modern physics. The theoretical investigation and experimental examination of Lorentz symmetry have witnessed considerable progress and attracted a lot of attentions since the mid of 1990s. It is inevitable to encounter quantum gravity in the exploration of the theoretical framework of high energy physics, especially around the energy scale near Planck scale. Different quantum gravity models neither exclude Lorentz violation nor predict it conclusively. Some high energy models of spacetime structure, such as non-commutative field theory, do, however, explicitly contain Lorentz violation. So the possible Lorentz violation is an important theoretical question [1].

There are many attempts to investigate the possible Lorentz violation from theoretical aspect [2–6]. Because at low energy scales, parity P, charge conjugation C and time reversal T are individually good symmetries of nature while there is evidence of CP violation for higher energies, one may consider the possible failure of Poincaré symmetry at such high energy scales. One theoretical possibility is that the spacetime symmetry of all the observed physical phenomena might be some proper subgroups of the Lorentz group along with the spacetime translations only if these kinds of proper subgroups of Poincaré group incorporating with either of the discrete operations P, T, CP or CT, can be enlarged to the full Poincaré group. The Very Special Relativity (VSR) proposal by Cohen and Glashow is based on these smaller subgroups [7]. Cohen and Glashow argued that the local symmetry of physics might not need to be as large as Lorentz group except its proper subgroup, while the full symmetry restores to Poincaré group when discrete symme-

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try P, T, CP or CT enters. The Lorentz violation is thus connected with CP violation. Since CP violating effects in nature are small, it is possible that Lorentz-violating effects may be similarly small. They identified these VSR subgroups up to isomorphism as T(2) (2-dimensional translations) with generators $T_1 = K_x + J_y$ and $T_2 = K_y - J_x$, where $\boldsymbol{J}$ and $\boldsymbol{K}$ are the generators of rotations and boosts respectively, E(2) (3-parameter Euclidean motion) with generators T_1, T_2 and J_z, HOM(2) (3-parameter orientation preserving transformations) with generators T_1, T_2 and K_z and SIM(2) (4-parameter similitude group) with generators T_1, T_2, J_z and K_z. The semi-direct product of the SIM(2) group with the spacetime translation group gives an 8-dimensional subgroup of the Poincaré group called ISIM(2). The spurion strategy can also be applied to VSP. The invariant tensor for group E(2) can be a 4-vector $n = (1, 0, 0, 1)$ while the symmetry groups T(2) admits many invariant tensors. There is neither invariant tensors for HOM(2) and SIM(2) nor the local Lorentz symmetry breaking perturbative discription for either of these groups.

Inspired by the fact that Poincaré group admits the unique deformation into de Sitter group, Gibbons, Gomis and Pope find that the subgroup ISIM(2) considered by Cohen and Glashow admits a 2-parameter family of continuous deformations which may be viewed as a quantum corrections or the quantum gravity effect to the very special relativity, but none of these give rise to noncommutative translations analogous to those of the de Sitter deformation of the Poincaré group: space-time remains flat. Among the 2-parameter family of deformation of ISIM(2), they find that only a 1-parameter DISIM$_b$(2), the deformation of SIM(2), is physically acceptable [8]. The line element invariant under DISIM$_b$(2) is Lorentz violating and of Finsler type, $\mathrm{d}s^2 = \left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right)^{1-b}\left(n_\mu \mathrm{d}x^\mu\right)^{2b}$. The DISIM$_b$(2) invariant action for point particle and the wave equations for spin 0, $\frac{1}{2}$ and 1 are derived in their paper. The equation for spin 0 field is in general a nonlocal equation, since it involves fractional even irrational derivatives.

The cosmological principle is the foundation of the standard cosmological model, the ΛCDM model, which assumes that the universe is isotropic and homogeneous at large scales. However, there are evidences which challenge the standard ΛCDM model in the cosmological and astronomical observations. These observations can be summarised as CMB multipole alignments, QSO polarization alignment and large scale bulk flows along a preferred cosmological axis [9]. The Planck satellite has found deviations from isotropy (around 3σ) by the CMB anisotropy observations recently [10]. The Finslerian geometry is naturally employed to account for this kind of anisotropic spacetime structure. The extensions of Einstein gravity theory to Finslerian type of geometry of spacetime have been proposed in recent years [11,12]. The FRW like spacetime and the refinement to the Schwarzschild solution in these Finslerian gravity frameworks are investigated. The DISIM$_b$(2) invariant Finslerian metric by Gibbons, Gomis and Pope's deformed very special relativity approach is employed in the investigations of anisotropy in FRW like cosmology, which leads to the Lorentz violating cosmology acceleration, and Lorentz violation as the trigger of density inhomogeneities to the cosmological fluid [13,14]. It is investigated that Randers spacetime with local symmetry group TE(2) can possess local symmetry of the generic VSR proposed by Cohen and Glashow and the implication of wich to type Ia supernovae [15] .

In this paper we follow Gibbons-Gomis-Pope's approach on the deformation of ISIM(2) and investigate the deformation of all such kinds of subgroups of Poincaré group which are the semi-product of three generators and four generators Lorentz subgroups with the spacetime translation group T(4) (semi-product Poincaré subgroup) and the five dimensional representations, which are inherited from the five dimensional representation of Poincaré group, (the natural representation) of all the semi-product Poincaré subgroup as well as their deformed partners. We find that the deformation of semi-product Poincaré subgroup may have more than one families that are physically acceptable. There may be more than one inequivalent natural representations for one family of deformation of a specific Poincaré subgroup. Usually the deformation of the original Lorentz subgroup part causes the rotational operation an additional accompanied scale factor which is not reasonable, for we believe that the departure from Lorentz symmetry should be from boost rather than rotational operation. Anyhow, most deformed boost operations do indeed have an additional accompanied scale factors which will play a key role in the search for group action invariant geometry and construction of field theories in the spactime of the invariant geometry.

2 Deformation of Lie algebra

The deformed Lie algebra or Lie group is extensively investigated [16,17]. Let's give here a short review on the deformation of Lie algebra according to Gibbons-Gomis-Pope. For a Lie algebra with commutation relations,

$$\left[T_i, T_j\right] = C_{ij}^k T_k, \tag{1}$$

we suppose that the structure constants of deformed Lie algebra is of the form

$$\hat{C}_{ij}^k = C_{ij}^k + tA_{ij}^k + t^2 B_{ij}^k + \cdots \tag{2}$$

Here t represents the deformation parameter. The constrain on deformed structure constants from Jacobi identity

$$\left[\left[T_i, T_j\right], T_k\right] + \left[\left[T_j, T_k\right], T_i\right] + \left[\left[T_k, T_i\right], T_j\right] = 0 \tag{3}$$

has the form

$$\hat{C}_{l[k}^m \hat{C}_{ij]}^l = \hat{C}_{lk}^m \hat{C}_{ij}^l + \hat{C}_{li}^m \hat{C}_{jk}^l + \hat{C}_{lj}^m \hat{C}_{ki}^l = 0. \tag{4}$$

The expansion of deformed structure constant with the power of t yields

$$t\left(A_{l[k}^m C_{ij]}^l + C_{l[k}^m A_{ij]}^l\right)$$

$$+ t^2\left(A^m_{l[k}A^l_{ij]} + B^m_{l[k}C^l_{ij]} + C^m_{l[k}B^l_{ij]}\right) + \cdots = 0. \tag{5}$$

If there exists a family of deformed Lie algebra parametrized by a continuous variable t, there should be a group of constrained equations which arise from every power of t in the above equation, as:

$$A^m_{l[k}C^l_{ij]} + C^m_{l[k}A^l_{ij]} = 0, \tag{6}$$
$$A^m_{l[k}A^l_{ij]} + B^m_{l[k}C^l_{ij]} + C^m_{l[k}B^l_{ij]} = 0, \tag{7}$$

and etc.

To avoid trivial deformation which arises merely from a change of basis in the original Lie algebra, one demands that there should not be a transformation of basis of Lie algebra $S^{\upsilon}_{\mu} = \delta^{\upsilon}_{\mu} + t\phi^{\upsilon}_{\mu} + \cdots \in \mathrm{GL}(n,\mathbb{R})$, such that $\hat{C}^k_{ij} = S^k_c C^c_{ab}\left(S^{-1}\right)^a_i\left(S^{-1}\right)^b_j$ and hence

$$A^k_{ij} = \phi^k_l C^l_{ij} - C^k_{lj}\phi^l_i - C^k_{il}\phi^l_j. \tag{8}$$

Define λ^{μ} as the basis vector of the original Lie algebra (the left invariant 1-form), then $d\lambda^i = -\frac{1}{2}C^i_{ab}\lambda^a \wedge \lambda^b$ [1,7]. We can define the vector valued one form field $\Phi^a = \phi^a_b\lambda^b$ and 2-form field $A^a = \frac{1}{2}A^a_{ij}\lambda^i \wedge \lambda^j$ and $B^a = \frac{1}{2}B^a_{ij}\lambda^i \wedge \lambda^j$ as well as a matrix valued 1-form field $C^b_a = \lambda^c C^b_{ca}$. So we have the covariant exterior differential operator of the present Lie algebra $D = d + C\wedge$, the formula (6) can be rewritten as:

$$DA^a = 0, A^a \neq -D\Phi^a. \tag{9}$$

The Jacobi Identity requires $D^2 = 0$, then

$$DB^a + (A \bullet A)^a = 0, \tag{10}$$

where $(A \bullet A)^a = \frac{1}{2}A^a_{b[c}A^b_{de]}\lambda^c \wedge \lambda^d \wedge \lambda^e$. The equation is solvale if $D(A \bullet A)^a = 0$.

If we set $A \bullet A = 0$, we find that the second order term of deformation will also satisfy eq. (9). Then the acceptable form of B^{μ} is the same as one of A^{μ}. It is enough to consider the first order deformed term only.

3 The proper subgroups of Lorentz group

The Lorentz Lie algebra has the following Lie sub-algebras up to isomorphism.

- Lie subalgebra with a single generator.
- Two Lie subalgebras with two generators: span$\{r_x, b_x\}$ and span$\left\{r_x + b_y, b_z\right\}$. The corresponding commutation relations are
  - span$\{r_x, b_x\}$: $[r_x, b_x] = 0$.
  - span$\left\{r_x + b_y, b_z\right\}$: $\left[b_x + r_y, b_z\right] = b_x + r_y$.
- Four Lie subalgebras with three generators: span$\left\{r_x, r_y, r_z\right\}$, span$\left\{b_x, b_y, r_z\right\}$, span$\{t_1, t_2, r_z\}$ and span$\{t_1, t_2, b_z\}$, where $t_1 = b_x + r_y$ and $t_2 = b_y - r_x$. The corresponding commutation relations are
  - span$\left\{r_x, r_y, r_z\right\}$ (the so(3)): $\left[r_x, r_y\right] = r_z$, $\left[r_y, r_z\right] = r_x$, $[r_z, r_x] = r_y$.
  - span$\left\{b_x, b_y, r_z\right\}$ (the Lorentz algebra in 2+1 dimension): $\left[b_x, b_y\right] = -r_z$, $\left[b_y, r_z\right] = b_x$, $[r_z, b_x] = b_y$.
  - span$\{t_1, t_2, r_z\}$ (the 2 dimensional Eudlidean algebra e(2)): $[t_1, t_2] = 0$, $[r_z, t_1] = t_2$, $[r_z, t_2] = -t_1$.
  - span$\{t_1, t_2, b_z\}$ (2-dimensional orientation preserving transformations group HOM(2)): $[t_1, t_2] = 0$, $[b_z, t_1] = -t_1$, $[b_z, t_2] = -t_2$.
- One Lie subalgebras with four generators: span$\{t_1, t_2, r_z, b_z\}$ (the 2 dimensional similitude group SIM(2)) with commutation relations $[t_1, t_2] = [r_z, b_z] = 0$, $[r_z, t_1] = t_2$, $[r_z, t_2] = -t_1$ and $[b_z, t_1] = -t_1$, $[b_z, t_2] = -t_2$.

The Lie subalgebra span$\{r_x, b_x\}$ is isomorphic to $\boldsymbol{t}(2) =$ span$\{t_1, t_2\}$, and they are both isomorphic to the 2 dimensional translation group T(2).

We will call the subgroup of Lorentz or Poincaré group as Lorentz or Poincaré subgroup for brevity.

4 The deformation group of the semi-product subgroups of Poincaré group

Poincaré group is the semi-direct product of Lorentz group with the translation group. Lorantz group is the normal subgroup of the Poincaré group which is generated by six generators, three rotation generators r_x, r_y, r_z and three boost generators b_x, b_y, b_z. The semi-direct product of subgroup of Lorentz group with translation group is also the subgroup of Poincaré group, which makes up one type of Poincaré subgroups. We will concentrate our attention on this type of subgroups and it is this type of Poincaré subgroup that Cohen and Glashow employ in their very special relativity proposal. The deformation groups of this type of subgroups can also be divided into two kinds. One kind consists of the semi-direct product of the deformation of Lorentz subgroup SL with T(4), which is regarded as the locally deformed group, while the deformation group of the other kind does not possess the semi-direct product structure, which is regarded as the globally deformed group. Among the globally deformed groups, the Lorentz subgroup does not deform in the first class but it will deform in the second class. We will concentrate on the first class of globally deformed groups, in which the deformation part comes from the intercrossing between Lorentz subgroup and the translational group and the translational group itself. The deformed group thus obtained does not have the semi-direct product structure of the Lorentz subgroup with the deformed translation group. Similar to the decomposition of Poincaré group into the Lorentz group, the local symmetry group, and the translational group which connect the local properties within a neighborhood, the deformed Poincaré subgroups can also be decomposed into two parts, one describe the local properties of the spacetime and the other part reflects the global properties of the spacetime to some extent. We mainly concentrate our attention on that kind of deformed Poincaré subgroups in which the Lorentz subgroup part is not deformed so that the local property of spacetime is the same as described in VSR.

From eq. (10), we obtained a constrain condition

$$D(A \bullet A)^a = 0. \tag{11}$$

The simple solution

$$A \bullet A = 0 \tag{12}$$

is a solution that satisfies all the constrain condition at all nonlinear orders. Then the constrain conditions from Jacobi Identity (10) can be written as:

$$DB^a = 0, \tag{13}$$

i.e., the second order deformation of structure constants B satisfies the same equation as A. Therefore we can get the higher order of deformation of structure constants in this way. Due to the simplest solution of the constrain condition (10) the deformation of the same group can have several different forms, e.g., the deformation group of IHOM, the semidirect product of HOM and T(4), has two different families. Of course the Poincaré group itself and ISIM group have only one family of deformation.

4.1 The perturbative solution of the representation of the deformed generators

The natural representation of the deformed generators can be viewed as some kind of perturbation of the representation of original group which inherits from the Poincaré group 5 dimensional natural matrix representation, for the deformed group can be viewed as the perturbation of the original group. The generators of deformed group can be written as $\{T'_i = T_i + \tau G_i\}$ and the corresponding structure constants as $C'^k_{ij} = C^k_{ij} + tA^k_{ij}$, where $\{T_i\}$ and C^k_{ij} are the generators and structure constants of the original group, hence

$$C^k_{ij} T_k = \left[T_i, T_j\right] \tag{14}$$

and

$$C'^k_{ij} T'_k = \left[T'_i, T'_j\right], \tag{15}$$

i.e.,

$$\tau^2 \left[G_i, G_j\right] + \tau\left(\left[G_i, T_j\right] + \left[T_i, G_j\right] - C^k_{ij} G_k - tA^k_{ij} G_k\right) - tA^k_{ij} T_k = 0, \tag{16}$$

where the generators Ts and Gs are all 5×5 matrices and the matrix elements of the unknown Gs are functions of the deformation parameter t. Moreover all of Gs are zero matrices when $t = 0$. We have now $N \times 5 \times 5 = 25N$ unknown variables for a Lie algebra with N generators, e.g., there are 250 unknown variables for Poincaré group, 200 for ISIM group and 175 for IHOM group respectively.

We can solve eq. (16) perturbatively. The dominant part of perturbation parameter τ for generators and t for structure constants should be in the same order. In general, we can assume that $tA^k_{ij} = \tau \bar{A}^k_{ij}$. Eq. (16) becomes

$$\begin{cases} \left[G_i, G_j\right] - \bar{A}^k_{ij} G_k = 0, \\ \left[G_i, T_j\right] + \left[T_i, G_j\right] - C^k_{ij} G_k - \bar{A}^k_{ij} T_k = 0. \end{cases} \tag{17}$$

The simplest case is $t_1 A^k_{ij} = \bar{A}^k_{ij}$ and $t = t_1 \tau$. Rewrite t_1 as t, and finally we have

$$\begin{cases} \left[G_i, G_j\right] - tA^k_{ij} G_k = 0, \\ \left[G_i, T_j\right] + \left[T_i, G_j\right] - C^k_{ij} G_k - tA^k_{ij} T_k = 0. \end{cases} \tag{18}$$

There may be more than one set of solutions due to the quadratic equations. We find that there may be more than one inequivalent natural representations for the deformation of a specific Lie algebra, which corresponds to different spacetime geometry.

4.2 The deformation of Poincaré group

The commutation relations for Poincaré group are

$$\begin{aligned} &\left[r_i, r_j\right] = \sum_{k=1}^{3} \varepsilon_{ijk} r_k, \left[b_i, b_j\right] = -\sum_{k=1}^{3} \varepsilon_{ijk} r_k, \\ &\left[b_i, r_j\right] = \sum_{k=1}^{3} \varepsilon_{ijk} b_k, \\ &\left[p_i, p_j\right] = 0, [r_i, p_t] = 0, \left[r_i, p_j\right] \\ &\quad = \sum_{k=1}^{3} \varepsilon_{ijk} p_k, [b_i, p_t] = p_i, \ \left[b_i, p_j\right] = \delta_{ij} p_t. \end{aligned} \tag{19}$$

The first order Jacobi constrain equation,

$$A^m_{l[k} C^l{}_{ij]} + C^m_{l[k} A^l{}_{ij]} = 0,$$

the simplest solution $A \bullet A = 0$ as the second order constrain, and the non-triviality condition,

$$A^k_{ij} \neq \phi^k_l C^l_{ij} - C^k_{lj} \phi^l_i - C^k_{il} \phi^l_j,$$

reduce most of the possible $10 \times \frac{10 \times 9}{2} = 450$ deformation parameters A^i_{jk} to zero and it can be verified that the deformation group of Poincaré group is unique and possesses the commutation structure,

$$\begin{aligned} &\left[r_i, r_j\right] = \sum_{k=1}^{3} \varepsilon_{ijk} r_k, \ \left[b_i, b_j\right] = -\sum_{k=1}^{3} \varepsilon_{ijk} r_k, \\ &\left[b_i, r_j\right] = \sum_{k=1}^{3} \varepsilon_{ijk} b_k, \ [r_i, p_t] = 0, \\ &\left[r_i, p_j\right] = \sum_{k=1}^{3} \varepsilon_{ijk} p_k, \ [b_i, p_t] = p_i, \ \left[b_i, p_j\right] = \delta_{ij} p_t, \\ &[p_t, p_i] = tb_i, \ \left[p_i, p_j\right] = -t \sum_{k=1}^{3} \varepsilon_{ijk} r_k, \end{aligned} \tag{20}$$

which is known as Lie algebra of de Sitter group.

The natural representation of the generators is also unique, which has the form,

$$p_t = \begin{pmatrix} &&&&1\\ &&&&\\ &&&&\\ &&&&\\ -t&&&& \end{pmatrix},\ p_x = \begin{pmatrix} &&&&\\ &&&&1\\ &&&&\\ &&&&\\ &t&&& \end{pmatrix},$$
$$p_y = \begin{pmatrix} &&&&\\ &&&&\\ &&&&1\\ &&&&\\ &&t&& \end{pmatrix},\ p_z = \begin{pmatrix} &&&&\\ &&&&\\ &&&&\\ &&&&1\\ &&&t& \end{pmatrix}, \tag{21}$$

where we only denote the non-zero matrix elements of the deformed generators, i.e., the representation matrix of the other six generators, the generators of the Lorentz group, remain unchanged.

4.3 The deformation of ISIM

The algebraic structure of ISIM, the semi-product of SIM with $T(4)$, is

$$\begin{aligned}
&[t_1, r_z] = -t_2,\ [t_1, b_z] = t_1,\ [t_1, p_t] = [t_1, p_z] = p_x,\\
&[t_2, r_z] = t_1,\ [t_2, b_z] = t_2,\ [t_2, p_t] = [t_2, p_z] = p_y,\\
&[t_1, p_x] = p_t - p_z,\ \left[t_2, p_y\right] = p_t - p_z,\ [r_z, p_x] = p_y,\\
&\left[r_z, p_y\right] = -p_x,\ [b_z, p_t] = p_z,\ [b_z, p_z] = p_t.
\end{aligned} \tag{22}$$

The Jacobi constrain reduces the $8 \times \frac{8\times 7}{2} = 224$ deformation parameters of the deformed group DISIM to 57. The simplest solution $A \bullet A = 0$ then is reduced further to 6 ones,

$$A^1_{1b},\ A^t_{1x},\ A^z_{1x},\ A^t_{rt},\ A^t_{bt},\ A^z_{bt}, \tag{23}$$

where $r,\ b,\ t,\ x,\ z$ represent $r_z,\ b_z,\ p_t,\ p_x,\ p_z$ respectively. The commutation relation for DISIM is

$$\begin{aligned}
&[t_1, r_z] = -t_2,\ [t_1, b_z] = \left(1 + A^1_{1b}\right) t_1,\ [t_2, r_z] = t_1,\ [t_1, p_t] = p_x,\ [t_1, p_x] = \left(1 + A^t_{1x}\right) p_t - \left(1 - A^z_{1x}\right) p_z,\\
&[t_1, p_z] = \left(1 + A^t_{1x} + A^z_{1x}\right) p_x,\ [t_2, b_z] = \left(1 + A^1_{1b}\right) t_2,\ [t_2, p_t] = p_y,\ \left[t_2, p_y\right] = \left(1 + A^t_{1x}\right) p_t - \left(1 - A^z_{1x}\right) p_z,\\
&[t_2, p_z] = \left(1 + A^t_{1x} + A^z_{1x}\right) p_y,\ [r_z, p_t] = A^t_{rt} p_t,\ [r_z, p_x] = p_y + A^t_{rt} p_x,\ \left[r_z, p_y\right] = -p_x + A^t_{rt} p_y,\\
&[r_z, p_z] = A^t_{rt} p_z,\ [b_z, p_x] = \left(A^t_{1x} + A^z_{1x} + A^t_{bt} + A^z_{bt} - A^1_{1b}\right) p_x,\ \left[b_z, p_y\right] = \left(A^t_{1x} + A^z_{1x} + A^t_{bt} + A^z_{bt} - A^1_{1b}\right) p_y,\\
&[b_z, p_t] = p_z + A^t_{bt} p_t + A^z_{bt} p_z,\ [b_z, p_z] = p_t + \left(2A^1_{1b} - A^z_{bt}\right) p_t + \left(2A^t_{1x} + 2A^z_{1x} + A^t_{bt} + 2A^z_{bt} - 2A^1_{1b}\right) p_z.
\end{aligned} \tag{24}$$

The non-triviality condition is

$$A^{t\ 2}_{rt} + \left(A^t_{1x} + A^z_{1x} + A^t_{bt} + A^z_{bt} - A^1_{1b}\right)^2 \neq 0. \tag{25}$$

The simplest solution $A \bullet A = 0$ gives

$$\begin{cases}
A^z_{1x}\left(A^t_{1x} + A^z_{1x}\right) = 0,\\
A^z_{bt}\left(A^t_{1x} + A^z_{1x}\right) = 0,\\
\left(A^t_{1x} - 2A^1_{1b}\right)\left(A^t_{1x} + A^z_{1x}\right) = 0.
\end{cases} \tag{26}$$

The existence of deformation parameter A^1_{1b} reveals that there is deformation inside of the original *sim* Lie subalgebra. We thus can specify DISIM into two families.

4.3.1 The deformation group with SIM undeformed

If $A^1_{1b} = 0$, SIM is undeformed in DISIM from eq. (24), the non-triviality condition now reads

$$A^{t\ 2}_{rt} + \left(A^t_{1x} + A^z_{1x} + A^t_{bt} + A^z_{bt}\right)^2 \neq 0. \tag{27}$$

The quadratic constrain condition becomes

$$\begin{cases}
A^z_{1x}\left(A^t_{1x} + A^z_{1x}\right) = 0,\\
A^z_{bt}\left(A^t_{1x} + A^z_{1x}\right) = 0,\\
A^t_{1x}\left(A^t_{1x} + A^z_{1x}\right) = 0.
\end{cases} \tag{28}$$

From eq. (28), the deformation group with SIM undeformed can be classified into two subfamilies: 1, $A^z_{1x} = -A^t_{1x}$, and 2, $A^z_{1x} = A^z_{bt} = A^t_{1x} = 0$.

In the first subfamily, A^t_{1x} can be absorbed into the redefinition of the generators,

$$\begin{cases}
t_i \to \left(1 + A^t_{1x}\right)^{-1/2} t_i,\ i = 1, 2,\\
p_\alpha \to \left(1 + A^t_{1x}\right)^{1/2} p_\alpha,\ \alpha = t, z.
\end{cases} \tag{29}$$

There are three deformation parameters left, $A^t_{rt},\ A^t_{bt},\ A^z_{bt}$, which can be simplified further. In fact, any A^t_{bt} gives the same Lie algebra up to an isomorphism when $A^t_{bt} + A^z_{bt}$ is kept fixed. For example, there are two Lie algebras, $t^{(i)}_1,\ t^{(i)}_2$, $r^{(i)}_z,\ b^{(i)}_z,\ p^{(i)}_t,\ p^{(i)}_x,\ p^{(i)}_y,\ p^{(i)}_z$ where $i = 1$ corresponds to one set of deformation parameters $A^t_{rt},\ A^t_{bt},\ A^z_{bt}$ and $i = 2$ corresponds to the other set of deformation parameters $A^t_{rt},\ B^t_{bt},\ B^z_{bt}$ satisfying $A^t_{bt} + A^z_{bt} = B^t_{bt} + B^z_{bt}$. We then can define

$$\begin{cases}
p^{(2)}_t = p^{(1)}_t + \frac{1}{2}\left(A^t_{bt} - B^t_{bt}\right)\left(p^{(1)}_t - p^{(1)}_z\right),\\
p^{(2)}_z = p^{(1)}_z + \frac{1}{2}\left(A^t_{bt} - B^t_{bt}\right)\left(p^{(1)}_t - p^{(1)}_z\right),
\end{cases} \tag{30}$$

such that $p^{(2)}_t - p^{(2)}_z = p^{(1)}_t - p^{(1)}_z$ and

$$\begin{aligned}
\left[b_z, p^{(2)}_t\right] &= \left[b_z, p^{(1)}_t\right] + \frac{A^t_{bt} - B^t_{bt}}{2}\left[b_z, p^{(1)}_t - p^{(1)}_z\right]\\
&= p^{(1)}_z + A^t_{bt} p^{(1)}_t + A^z_{bt} p^{(1)}_z\\
&\quad + \frac{A^t_{bt} - B^t_{bt}}{2}\left(-1 + A^t_{bt} + A^z_{bt}\right)\left(p^{(1)}_t - p^{(1)}_z\right)\\
&= p^{(1)}_z - \frac{A^t_{bt} - B^t_{bt}}{2}\left(p^{(1)}_t - p^{(1)}_z\right) + A^t_{bt} p^{(2)}_t + A^z_{bt} p^{(2)}_z\\
&= p^{(2)}_z - \left(A^t_{bt} - B^t_{bt}\right)\left(p^{(2)}_t - p^{(2)}_z\right) + A^t_{bt} p^{(2)}_t + A^z_{bt} p^{(2)}_z\\
&= p^{(2)}_z + B^t_{bt} p^{(2)}_t + \left(A^t_{bt} + A^z_{bt} - B^t_{bt}\right) p^{(2)}_z.
\end{aligned} \tag{31}$$

We therefore only consider two cases in which $A^t_{bt} = 0$ or $A^z_{bt} = 0$.

In the second subfamily, there are two deformation parameters A^t_{rt} and A^t_{bt}, and therefore it can be classified into the first subfamily.

There remain two cases to be investigated, $A^t_{bt} = 0$ for the first case and $A^z_{bt} = 0$ for the second case.

Let's consider the first case in which $A^t_{bt} = 0$. Denoting $A_1 = A^t_{rt}$ and $A_2 = A^t_{bt}$, the representation matrices of the deformed generators are

$$r_z = \begin{pmatrix} A_1 & & & & \\ & A_1 & -1 & & \\ & 1 & A_1 & & \\ & & & A_1 & \\ & & & & 0 \end{pmatrix}, \quad b_z = \begin{pmatrix} A_2 & & & 1 & \\ & A_2 & & & \\ & & A_2 & & \\ 1 & & & A_2 & \\ & & & & 0 \end{pmatrix}, \tag{32}$$

and the corresponding single parameter group elements are

$$\begin{aligned} R_z(\theta) &= \begin{pmatrix} e^{\theta A_1} & & & & \\ & e^{\theta A_1}\cos\theta & -e^{\theta A_1}\sin\theta & & \\ & e^{\theta A_1}\sin\theta & e^{\theta A_1}\cos\theta & & \\ & & & e^{\theta A_1} & \\ & & & & 1 \end{pmatrix}, \\ B_z(\theta) &= \begin{pmatrix} e^{\theta A_2}\cosh\theta & & & e^{\theta A_2}\sinh\theta & \\ & e^{\theta A_2} & & & \\ & & e^{\theta A_2} & & \\ e^{\theta A_2}\sinh\theta & & & e^{\theta A_2}\cosh\theta & \\ & & & & 1 \end{pmatrix}, \end{aligned} \tag{33}$$

where the deformed rotation $R_z(\theta)$ is not a merely rotation anymore but a rotation followed by a dilatation $e^{\theta A_1}$. $R_z(2\pi) = e^{2\pi A_1}$ is a pure dilatation when $A_1 \neq 0$. To keep $R_z(\theta)$ as a reasonable local rotation operation, one demands $A_1 = 0$. There survives only one deformation parameter A_2, denoted by b hereafter, for this case. The representation matrix of the deformed boost operation is now of the form,

$$B_z(\theta) = e^{b\theta} \begin{pmatrix} \cosh\theta & & & \sinh\theta \\ & 1 & & \\ & & 1 & \\ \sinh\theta & & & \cosh\theta \end{pmatrix}, \tag{34}$$

an ordinary boost followed by a dilatation.

In the second case, $A^z_{bt} = 0$. Denoting $A_1 = A^t_{rt}$ and $A_2 = A^z_{bt}$, what is different from the first case just investigated is that there may be a group of matrix representation for the deformed group which is specified by a free parameter λ:

$$\begin{aligned} r_z &= \begin{pmatrix} A_1 & & & & \\ & A_1 & -1 & & \\ & 1 & A_1 & & \\ & & & A_1 & \\ & & & & 0 \end{pmatrix}, \\ b_z &= \begin{pmatrix} 2\lambda & & & 1 - A_2 + 2\lambda & \\ & A_2 & & & \\ & & A_2 & & \\ 1 + A_2 - 2\lambda & & & 2(A_2 - \lambda) & \\ & & & & 0 \end{pmatrix}, \\ p_t &= \begin{pmatrix} 0 & & & & 1 + \lambda \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -\lambda \\ & & & & 0 \end{pmatrix}, \quad p_z = \begin{pmatrix} 0 & & & & \lambda \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 1 - \lambda \\ & & & & 0 \end{pmatrix}. \end{aligned} \tag{35}$$

Similar to the first case, one can arrive at a reasonable local rotation operation by forcing the rotation generator undeformed. The free parameter λ actually represents the choice of coordinate system. This means that the representation matrices which different λ corresponds to can be transformed from one to another by a coordinate transformation, e.g., the matrix representation of $\lambda = \lambda_1$ can be transformed into ones of $\lambda = \lambda_2$ by the following coordinate transformation matrix,

$$T = \begin{pmatrix} 1 - \lambda_1 + \lambda_2 & & & \lambda_2 - \lambda_2 & \\ & 1 & & & \\ & & 1 & & \\ \lambda_1 - \lambda_2 & & & 1 + \lambda_1 - \lambda_2 & \\ & & & & 1 \end{pmatrix}. \tag{36}$$

What we need is therefore to choose an appropriate λ, e.g., $\lambda = \frac{A_2}{2}$, and the representation matrices for generators are

$$\begin{aligned} b_z &= \begin{pmatrix} A_2 & & & 1 & \\ & A_2 & & & \\ & & A_2 & & \\ 1 & & & A_2 & \\ & & & & 0 \end{pmatrix}, \quad p_t = \begin{pmatrix} 0 & & & & 1 + \frac{A_2}{2} \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -\frac{A_2}{2} \\ & & & & 0 \end{pmatrix}, \\ p_t &= \begin{pmatrix} 0 & & & & \frac{A_2}{2} \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 1 - \frac{A_2}{2} \\ & & & & 0 \end{pmatrix}. \end{aligned} \tag{37}$$

The corresponding single parameter group elements are

$$\begin{aligned} B_z(\theta) &= e^{b\theta} \begin{pmatrix} \cosh\theta & & & \sinh\theta \\ & 1 & & \\ & & 1 & \\ \sinh\theta & & & \cosh\theta \end{pmatrix}, \\ P_t(\lambda) &= \begin{pmatrix} \lambda + \frac{A_2}{2}\lambda \\ \\ \\ -\frac{A_2}{2}\lambda \end{pmatrix}, \quad P_z(\lambda) = \begin{pmatrix} \frac{A_2}{2}\lambda \\ \\ \\ \lambda - \frac{A_2}{2}\lambda \end{pmatrix}. \end{aligned} \tag{38}$$

Note that there are many different matrix representations as a matter of fact. However, the 5×5 representation matrices of the deformed group elements have their origin from the 5×5 representation of Poincaré group which has a special geometric explanation. The 5×5 representation of the deformed group should have the same geometric explanation, i.e., the upper left 4×4 part of the representation matrix represents rotation and boost, the upper right 1×4 part represents translation and the lower 5×1 part should be kept zero. The following matrix representation of the first subclass is excluded with this restriction,

$$r_z = \begin{pmatrix} 0 & & & & \\ & 0 & -1 & & \\ & 1 & 0 & & \\ & & & 0 & \\ & & & & -A_1 \end{pmatrix}, \quad b_z = \begin{pmatrix} 0 & & & 1 & \\ & 0 & & & \\ & & 0 & & \\ 1 & & & 0 & \\ & & & & -A_2 \end{pmatrix}, \tag{39}$$

which do not have an apparent geometric explanation. We will ignore such a kind of representation hereafter.

4.3.2 The deformation group with SIM deformed

In the last section we have investigated the deformation group in which the SIM part remains un-deformed and the corresponding natural representation. We are going to investigate the deformation of ISIM in which the SIM itself also deforms and the corresponding natural representation in this section. Like the case where SIM is undeformed, we can specify two subfamilies, 1. $A^z_{1x} = -A^t_{1x}$ and 2. $A^z_{1x} = A^z_{bt} = 0$ and $A^t_{1x} = 2A^1_{1b}$.

The first subfamily is denoted by *xdisim*1, in which there are 4 deform parameters, A^1_{1b}, A^t_{rt}, A^t_{bt}, A^z_{bt}, where the deformed Lie algebra with an arbitrary value of A^t_{bt} is the same one up to an isomorphism only if $A^t_{bt} + A^z_{bt}$ is kept fixed. There are three independent deform parameters, A^1_{1b}, A^t_{rt} and $A^t_{bt} + A^z_{bt}$ actually.

We can also specify two cases further as in the last section. In the first case , the independent deform parameters are A^1_{1b}, A^t_{rt} and A^t_{bt}, and the commutation relations are

$$\begin{aligned}&[t_1,b_z]=\left(1+A^1_{1b}\right)t_1,\ [t_2,b_z]=\left(1+A^1_{1b}\right)t_2,\ [r_z,p_t]=A^t_{rt}p_t,\ [r_z,p_x]=p_y+A^t_{rt}p_x,\ \left[r_z,p_y\right]=-p_x+A^t_{rt}p_y,\ [r_z,p_z]=A^t_{rt}p_z,\\ &[b_z,p_t]=p_z+A^t_{bt}p_t,\ [b_z,p_x]=\left(A^t_{bt}-A^1_{1b}\right)p_x,\ \left[b_z,p_y\right]=\left(A^t_{bt}-A^1_{1b}\right)p_y,\ [b_z,p_z]=p_t+2A^1_{1b}p_t+\left(A^t_{bt}-2A^1_{1b}\right)p_z.\end{aligned} \tag{40}$$

The natural matrix representation are

$$\begin{aligned}&b_z=\begin{pmatrix}\alpha-2A_1+A_3&&&1+\alpha\\&A_3-A_1&&\\&&A_3-A_1&\\1-\alpha+2A_1&&&A_3-\alpha\end{pmatrix},\ r_z=\begin{pmatrix}A_2&&&\\&A_2&-1&\\&1&A_2&\\&&&A_2\end{pmatrix},\\ &p_t=\begin{pmatrix}1+\frac{\alpha}{2}\\ \\ \\A_1-\frac{\alpha}{2}\end{pmatrix},\ p_x=\begin{pmatrix}\\1+A_1\\ \\ \\ \end{pmatrix},\ p_y=\begin{pmatrix}\\ \\1+A_1\\ \\ \end{pmatrix},\ p_z=\begin{pmatrix}\frac{\alpha}{2}-A_1\\ \\ \\1+2A_1-\frac{\alpha}{2}\end{pmatrix},\end{aligned} \tag{41}$$

where α is a free parameter such that the matrix representations of different values can be transformed from one to another. The transformation matrix

$$T=\begin{pmatrix}1+\frac{\alpha_2-\alpha_1}{2+2A_1}&&&\frac{\alpha_2-\alpha_1}{2+2A_1}&\\&1&&&\\&&1&&\\-\frac{\alpha_2-\alpha_1}{2+2A_1}&&&1-\frac{\alpha_2-\alpha_1}{2+2A_1}&\\&&&&1\end{pmatrix} \tag{42}$$

can transform the matrix representation of $\alpha=\alpha_1$ to one of $\alpha=\alpha_2$. We give α a suitable value, e.g., $\alpha=A_1$ and therefore

$$\begin{aligned}&b_z=\begin{pmatrix}A_3-A_1&&&1+A_1\\&A_3-A_1&&\\&&A_3-A_1&\\1+A_1&&&A_3-A_1\end{pmatrix},\ r_z=\begin{pmatrix}A_2&&&\\&A_2&-1&\\&1&A_2&\\&&&A_2\end{pmatrix},\\ &p_t=\begin{pmatrix}1+\frac{A_1}{2}\\ \\ \\\frac{A_1}{2}\end{pmatrix},\ p_x=\begin{pmatrix}\\1+A_1\\ \\ \\ \end{pmatrix},\ p_y=\begin{pmatrix}\\ \\1+A_1\\ \\ \end{pmatrix},\ p_z=\begin{pmatrix}-\frac{A_1}{2}\\ \\ \\1+\frac{3A_1}{2}\end{pmatrix}.\end{aligned} \tag{43}$$

The corresponding single parameter group elements are

$$B_z(\theta)=\mathrm{e}^{\theta(A_3-A_1)}\begin{pmatrix}\cosh(1+A_1)\theta&&&\sinh(1+A_1)\theta\\&1&&\\&&1&\\\sinh(1+A_1)\theta&&&\cosh(1+A_1)\theta\end{pmatrix},\ P_t(\lambda)=\begin{pmatrix}\lambda+\frac{A_1}{2}\lambda\\ \\ \\\frac{A_1}{2}\lambda\end{pmatrix},\ P_z(\lambda)=\begin{pmatrix}-\frac{A_1}{2}\lambda\\ \\ \\\lambda+\frac{3A_1}{2}\lambda\end{pmatrix}. \tag{44}$$

In the second case of xdisim1, the deform parameters are A^1_{1b}, A^t_{rt}, A^z_{bt} and the commutation relations are

$$\begin{aligned}&[t_1,b_z]=\left(1+A^1_{1b}\right)t_1,\ [t_2,b_z]=\left(1+A^1_{1b}\right)t_2,\ [r_z,p_t]=A^t_{rt}p_t,\ [r_z,p_x]=p_y+A^t_{rt}p_x,\\ &\left[r_z,p_y\right]=-p_x+A^t_{rt}p_y,\ [r_z,p_z]=A^t_{rt}p_z,\ [b_z,p_t]=p_z+A^z_{bt}p_z,\ [b_z,p_x]=\left(A^z_{bt}-A^1_{1b}\right)p_x,\\ &\left[b_z,p_y\right]=\left(A^z_{bt}-A^1_{1b}\right)p_y,\ [b_z,p_z]=p_t+\left(2A^1_{1b}-A^z_{bt}\right)p_t+2\left(A^z_{bt}-A^1_{1b}\right)p_z.\end{aligned} \tag{45}$$

There are many equivalent representations and we can choose a simple one as in the first case,

$$r_z=\begin{pmatrix}A_2&&&&\\&A_2&-1&&\\&1&A_2&&\\&&&A_2&\\&&&&0\end{pmatrix},\ b_z=\begin{pmatrix}0&&&1-A_3+2A_1&\\&A_3-A_1&&&\\&&A_3-A_1&&\\1+A_3&&&2(A_3-A_1)&\\&&&&0\end{pmatrix}, \tag{46}$$

where A_i represents $A^1_{1b},\ A^t_{rt},\ A^z_{bt}$. For the same reason as in the last section, it is required that a local rotation operation should not have an additional dilatation transformation constrains $A_2 = 0$. Hence the deformed group element is

$$B_z(\theta) = \mathrm{e}^{\theta(A_3-A_1)} \begin{pmatrix} \cosh\omega + \frac{A_1-A_3}{1+A_1}\sinh\omega & & & \frac{1+2A_1-A_3}{1+A_1}\sinh\omega \\ & 1 & & \\ & & 1 & \\ \frac{1+A_3}{1+A_1}\sinh\omega & & & \cosh\omega - \frac{A_1-A_3}{1+A_1}\sinh\omega \end{pmatrix}, \tag{47}$$

where $\omega = (1+A_1)\,\theta$. Note that the boost operation does not have an additional accompanied dilatation operation when $A_3 = A_1$.

The second subfamily is denoted by xdisim2, in which there remain three deform parameters, $A^1_{1b},\ A^t_{rt},\ A^t_{bt}$, for $A^z_{1x} = A^z_{bt} = 0,\ A^t_{1x} = 2A^1_{1b}$, and the commutation relations become

$$\begin{aligned} &[t_1, b_z] = \left(1 + A^1_{1b}\right) t_1,\ \ [t_2, b_z] = \left(1 + A^1_{1b}\right) t_2,\ \ [t_1, p_x] = \left(1 + 2A^1_{1b}\right) p_t - p_z,\ \ [t_1, p_z] = \left(1 + 2A^1_{1b}\right) p_x, \\ &\left[t_2, p_y\right] = \left(1 + 2A^1_{1b}\right) p_t - p_z,\ \ [t_2, p_z] = \left(1 + 2A^1_{1b}\right) p_y,\ \ [r_z, p_t] = A^t_{rt} p_t,\ \ [r_z, p_x] = p_y + A^t_{rt} p_x, \\ &\left[r_z, p_y\right] = -p_x + A^t_{rt} p_y,\ \ [r_z, p_z] = A^t_{rt} p_z,\ \ [b_z, p_t] = p_z + A^t_{bt} p_t,\ \ [b_z, p_x] = \left(A^1_{1b} + A^t_{bt}\right) p_x, \\ &\left[b_z, p_y\right] = \left(A^1_{1b} + A^t_{bt}\right) p_y,\ \ [b_z, p_z] = \left(1 + 2A^1_{1b}\right) p_t + \left(2A^1_{1b} + A^t_{bt}\right) p_z. \end{aligned} \tag{48}$$

There are many equivalent natural representations of this deformed group, one of which is as follows,

$$r_z = \begin{pmatrix} A_2 & & & & \\ & A_2 & -1 & & \\ & 1 & A_2 & & \\ & & & A_2 & \\ & & & & 0 \end{pmatrix},\ b_z = \begin{pmatrix} 2A_1 + A_3 & & & 1 + 2A_1 & \\ & A_1 + A_3 & & & \\ & & A_1 + A_3 & & \\ 1 & & & A_3 & \\ & & & & 0 \end{pmatrix},\ p_z = \begin{pmatrix} 0 & & & & 2A_1 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}. \tag{49}$$

The deform parameter in the rotation generator is supposed to be zero for the same reason that we need a resonable local rotation operation. Now we arrive at the natural representation of the deformed single parameter group element,

$$B_z(\theta) = \mathrm{e}^{(A_1+A_3)\theta} \begin{pmatrix} \cosh\omega + \frac{A_1}{1+A_1}\sinh\omega & & & \frac{1+2A_1}{1+A_1}\sinh\omega \\ & 1 & & \\ & & 1 & \\ \frac{1}{1+A_1}\sinh\omega & & & \cosh\omega - \frac{A_1}{1+A_1}\sinh\omega \end{pmatrix},\ P_z(\lambda) = \begin{pmatrix} 2A_1\lambda \\ \\ \\ \lambda \end{pmatrix}, \tag{50}$$

where $\omega = (1 + A_1)\,\theta$. Note that the boost operation does not have the additional accompanied dilatation when $A_3 = -A_1$ similar to in the previous cases.

4.4 The deformation of IHOM

The Lie algebra of semi-direct product of HOM with T(4) has the following commutation relations,

$$\begin{aligned} &[t_1, b_z] = t_1,\ \ [t_1, p_t] = [t_1, p_z] = p_x,\ \ [t_1, p_x] = p_t - p_z, \\ &[t_2, b_z] = t_2,\ \ [t_2, p_t] = [t_2, p_z] = p_y, \\ &\left[t_2, p_y\right] = p_t - p_z,\ \ [b_z, p_t] = p_z,\ \ [b_z, p_z] = p_t. \end{aligned}$$

The deform group DIHOM of IHOM which keeps HOM undeformed has four deform parameters which satisfy three second order constrain conditions

$$A^1_{1y} A^b_{bt} = A^1_{1y} A^z_{bt} = A^b_{2t}\left(A^t_{bt} + A^z_{bt}\right) = 0,$$

and the non-triviality condition

$$\left(A^1_{1y} + A^b_{2t}\right)^2 + \left(A^t_{bt} + A^z_{bt}\right)^2 \neq 0.$$

It can be classified into two families. One is denoted by dihom1 with $A^1_{1y} = A^b_{2t} = 0$ and has two deform parameters $A^t_{bt},\ A^z_{bt}$, the other is denoted by dihom2 with $A^z_{bt} = -A^t_{bt}$ and has two deform parameters $A^1_{1y},\ A^b_{2t}$.

The commutation relations for dihom1 is

$$\begin{aligned} &[b_z, p_t] = p_z + A^t_{bt} p_t + A^z_{bt} p_z, \\ &[b_z, p_z] = p_t + \left(A^t_{bt} + 2A^z_{bt}\right) p_z - A^z_{bt} p_t, \\ &[b_z, p_x] = \left(A^t_{bt} + A^z_{bt}\right) p_x,\ \ \left[b_z, p_y\right] = \left(A^t_{bt} + A^z_{bt}\right) p_y. \end{aligned} \tag{51}$$

Note that any value of A^t_{bt} when $A^t_{bt} + A^z_{bt}$ is kept fixed gives the same deformed Lie algebra just as what happens in deformed sim lie algebra. We therefore take $A^z_{bt} = 0$. Note also that the commutation relation of dihom1 is almost the same as one of the deformed isim algebra with sim part invariant. The difference is that dihom1 has one less generators than disim. The deformed part of the natural representation of dihom1 is

$$b_z = \begin{pmatrix} A_1 & & & 1 & \\ & A_1 & & & \\ & & A_1 & & \\ 1 & & & A_1 & \\ & & & & 0 \end{pmatrix}, \tag{52}$$

which is apparently the same as in disim1. Taking $A^t_{bt} = 0$ is another choice and the natural representation of deformation part is

$$b_z = \begin{pmatrix} 0 & & & 1-A_1 & \\ & A_1 & & & \\ & & A_1 & & \\ 1+A_1 & & & 2A_1 & \\ & & & & 0 \end{pmatrix}, \tag{53}$$

which is the same as in disim2.

There is another deformation group DIHOM2 of IHOM, which is not isomorphic to DIHOM1 and its deformed part has the following commutation relations,

$$\begin{aligned} &[t_2, p_x] = \left(A^1_{1y} + A^b_{2t}\right) t_1, \ \ [t_2, p_t] = p_y + A^b_{2t} b_z, \\ &\left[t_2, p_y\right] = p_t - p_z + \left(2A^1_{1y} + A^b_{2t}\right) t_2, \ \ [t_2, p_z] = p_y + A^b_{2t} b_z, \\ &\left[t_1, p_y\right] = A^1_{1y} t_1, \ \ \left[p_y, p_t\right] = \left(A^1_{1y} + A^b_{2t}\right) p_t + A^1_{1y} p_z, \\ &\left[p_y, p_x\right] = \left(A^1_{1y} + A^b_{2t}\right) p_x, \ \ \left[p_y, p_z\right] = \left(A^1_{1y} + A^b_{2t}\right) p_z + A^1_{1y} p_t. \end{aligned} \tag{54}$$

The natural matrix representation therefore can be solved as:

$$\begin{aligned} &t_2 = \begin{pmatrix} & 1 & \\ & & \\ 1 & & 1 \\ & -1 & \\ -\delta & & -\delta \end{pmatrix}, \ b_z = \begin{pmatrix} \gamma & & & 1 & \\ & \gamma & & & \\ & & \gamma & & \\ 1 & & & \gamma & \\ & & & & \gamma \end{pmatrix}, \ p_t = \begin{pmatrix} & 1 \\ & \\ -\delta & \\ & \\ & \end{pmatrix}, \\ &p_x = \begin{pmatrix} & \\ & 1 \\ \delta & \\ & \\ & \end{pmatrix}, \ p_y = \begin{pmatrix} -\gamma A_2 & & & A_1 & \\ & -\gamma A_2 & & & \\ & & \delta - \gamma A_2 & & 1 \\ A_1 & & & -\gamma A_2 & \\ & & & & -\delta - \gamma A_2 \end{pmatrix}, \ p_z = \begin{pmatrix} & \\ & \\ \delta & \\ & 1 \\ & \end{pmatrix}, \end{aligned} \tag{55}$$

where γ is an arbitrary parameter and $\delta = A_1 + A_2$. Here we only list the matrices for deformed generators. Note that regardless of the value of γ, the $(5,5)$ element of either b_z or p_y is nonzero. Moreover, the 5th row of t_2 is non-zero. So the matrix representation of dihim2 is different from the ones of various deformed Lie algebra. The corresponding representation spacetime is apparently curved globally. Note also that the representation with different γ is inequivalent in general. Take $\gamma = 0$, and we have

$$t_2 = \begin{pmatrix} & 1 & \\ & & \\ 1 & & 1 \\ & -1 & \\ -\delta & & -\delta \end{pmatrix}, \ p_y = \begin{pmatrix} & & A_1 & \\ & & & \\ & \delta & & 1 \\ A_1 & & & \\ & & & -\delta \end{pmatrix}, \ p_t = \begin{pmatrix} & 1 \\ & \\ -\delta & \\ & \\ & \end{pmatrix}, \ p_x = \begin{pmatrix} & \\ & 1 \\ \delta & \\ & \\ & \end{pmatrix}, \ p_z = \begin{pmatrix} & \\ & \\ \delta & \\ & 1 \\ & \end{pmatrix}. \tag{56}$$

The representation of dihom2 is totally different from one of dihom1.

4.5 The deformed group of TE(2)

Just like HOM group, the E(2) group is also the subgroup of Lorentz group with three generators. The corresponding Lie algebra is e(2). The semiproduct of E(2) and T(4) is denoted by TE, and its Lie algebra is denoted by te with the commutation relations,

$$\begin{aligned} &[t_1, r_z] = -t_2, \ \ [t_1, p_t] = [t_1, p_z] = p_x, \\ &[t_1, p_x] = p_t - p_z, \ \ [t_2, r_z] = t_1, \\ &[t_2, p_t] = [t_2, p_z] = p_y, \ \ \left[t_2, p_y\right] = p_t - p_z, \\ &[r_z, p_x] = p_y, \ \ \left[r_z, p_y\right] = -p_x. \end{aligned} \tag{57}$$

The deformed TE is DTE with 5 deform parameters A^1_{1t}, A^t_{rt}, A^z_{rt}, A^1_{tx}, A^x_{tx} under the second order constrain conditions,

$$\begin{cases} A^1_{1t}\left(A^t_{rt} + A^z_{rt}\right) = 0, \ \ A^1_{tx}\left(A^t_{rt} + A^z_{rt}\right) = 0, \\ A^x_{tx}\left(A^t_{rt} + A^z_{rt}\right) = 0, \ \ A^1_{1t} A^x_{tx} = 0. \end{cases} \tag{58}$$

The non-triviality condition is

$$\left(A^t_{rt} + A^z_{rt}\right)^2 + \left(A^t_{rt} + 2A^z_{rt}\right)^2 + \left(A^1_{tx}\right)^2 + \left(A^1_{1t} - A^x_{tx}\right)^2 \neq 0. \tag{59}$$

The DTE therefore can be divided into several families similar to what happens in DISIM and DIHOM.

1. $A^t_{rt} + A^z_{rt} \neq 0$, $A^1_{1t} = A^1_{tx} = A^x_{tx} = 0$, the corresponding deformed Lie algebra is denoted by dte1 with the following commutation relations,

$$\begin{aligned} &[r_z, p_t] = A^t_{rt} p_t + A^z_{rt} p_z, \ \ [r_z, p_x] = p_y + \left(A^t_{rt} + A^z_{rt}\right) p_x, \\ &\left[r_z, p_y\right] = -p_x + \left(A^t_{rt} + A^z_{rt}\right) p_y, \\ &[r_z, p_z] = A^z_{rt} p_t + \left(A^t_{rt} + 2A^z_{rt}\right) p_z. \end{aligned} \tag{60}$$

The non-triviality condition is now

$$\left(A^t_{rt} + A^z_{rt}\right)^2 + \left(A^t_{rt} + 2A^z_{rt}\right)^2 + \left(A^t_{rt}\right)^2 + \left(A^z_{rt}\right)^2 \neq 0. \tag{61}$$

The matrix representation is

$$r_z = \begin{pmatrix} A_1 & & & -A_2 & \\ & A_1 + A_2 & -1 & & \\ & 1 & A_1 + A_2 & & \\ A_2 & & & A_1 + 2A_2 & \\ & & & & 0 \end{pmatrix}, \tag{62}$$

where $A_1 = A^t_{rt}$ and $A_2 = A^z_{rt}$, while the corresponding group element as:

$$R_z(\theta) = \mathrm{e}^{(A_1+A_2)\theta}\begin{pmatrix} 1-A_2\theta & & & -A_2\theta \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ A_2\theta & & & 1+A_2\theta \end{pmatrix}. \quad (63)$$

It is apparent that the rotation operation changes a lot and may have additional accompanied dilatation as in disim. Moreover, the rotation itself is not only a rotation in the xy plane but also rotation in the rotated tz plane.

2. $A^t_{rt} + A^z_{rt} = 0$, $A^1_{1t} = 0$, the corresponding Lie algebra is denoted by dte2. There are three deform parameters A^t_{rt}, A^1_{tx}, A^x_{tx} and the commutation relations are

$$\begin{aligned} &[r_z, p_t] = A^t_{rt}(p_t - p_z), \ [r_z, p_z] = A^t_{rt}(p_t - p_z), \ [p_t, p_x] = A^1_{tx}t_1 + A^x_{tx}p_x, \ \left[p_t, p_y\right] = A^1_{tx}t_2 + A^x_{tx}p_y, \\ &[p_t, p_z] = A^x_{tx}(p_z - p_t), \ [p_z, p_x] = A^1_{tx}t_1 + A^x_{tx}p_x, \ \left[p_z, p_y\right] = A^1_{tx}t_2 + A^x_{tx}p_y, \end{aligned} \quad (64)$$

which can be simplified further by a linear transformation in Lie algebra

$$T = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & \lambda & & & -\lambda \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}. \quad (65)$$

The new set of generators after T transformation differs from the old only with r_z replaced by $r'_z = r_z + \lambda(p_t - p_z)$. Then the new commutation relations are

$$\begin{aligned} &\left[t_1, r'_z\right] = [t_1, r_z] + \lambda([t_1, p_t] - [t_1, p_z]) = [t_1, r_z] - \lambda(p_x - p_x) = [t_1, r_z], \\ &\left[t_2, r'_z\right] = [t_2, r_z] + \lambda([t_2, p_t] - [t_2, p_z]) = [t_2, r_z] - \lambda\left(p_y - p_y\right) = [t_2, r_z], \\ &\left[r'_z, p_t\right] = [r_z, p_t] + \lambda[p_t, p_z] = A^t_{rt}(p_t - p_z) + \lambda A^x_{tx}(p_z - p_t) = (A^t_{rt} - \lambda A^x_{tx})(p_t - p_z), \\ &\left[r'_z, p_z\right] = [r_z, p_z] + \lambda[p_t, p_z] = A^t_{rt}(p_t - p_z) + \lambda A^x_{tx}(p_z - p_t) = (A^t_{rt} - \lambda A^x_{tx})(p_t - p_z), \\ &\left[r'_z, p_x\right] = [r_z, p_x] + \lambda([p_t, p_x] - [p_z, p_x]) = 0, \ \left[r'_z, p_y\right] = \left[r_z, p_y\right] + \lambda\left(\left[p_t, p_y\right] - \left[p_z, p_y\right]\right) = 0, \end{aligned} \quad (66)$$

i.e., the commutation relations are almost kept unchanged except $[r'_z, p_t]$ and $[r'_z, p_t]$. Define $A'^t_{rt} = A^t_{rt} - \lambda A^x_{tx}$, the new commutation relations are

$$\begin{cases} \left[r'_z, p_t\right] = A'^t_{rt}(p_t - p_z) = (A^t_{rt} - \lambda A^x_{tx})(p_t - p_z), \\ \left[r'_z, p_z\right] = A'^t_{rt}(p_t - p_z) = (A^t_{rt} - \lambda A^x_{tx})(p_t - p_z). \end{cases} \quad (67)$$

Hence A^t_{tx} and A^x_{tx} are not independent parameters. We can specify two subfamilies of deformation group DTE2 further. One subfamily is denoted by dte2a in which A^1_{tx}, A^x_{tx} are taken as the independent parameters. The deformed commutation relations are

$$\begin{aligned} &[p_t, p_x] = A^1_{tx}t_1 + A^x_{tx}p_x, \ \left[p_t, p_y\right] = A^1_{tx}t_2 + A^x_{tx}p_y, \\ &[p_t, p_z] = A^x_{tx}(p_z - p_t), \\ &[p_z, p_x] = A^1_{tx}t_1 + A^x_{tx}p_x, \ \left[p_z, p_y\right] = A^1_{tx}t_2 + A^x_{tx}p_y. \end{aligned} \quad (68)$$

In the perturbation expansion of its matrix representation, the first order of some $\overline{A}^k_{ij}$ in eq. (17) does not give contribution but their second order does, i.e., $A^1_{tx} = \alpha\tau^2$, $A^x_{tx} = \beta\tau$. There are two inequivalent representations for this subfamily. One is

$$\begin{aligned} &p_t = \begin{pmatrix} \alpha+2\beta & & & \alpha & 1 \\ & \beta & & & \\ & & \beta & & \\ -\alpha-A_2 & & & 2\beta-\alpha-A_2 & \\ & & & & 0 \end{pmatrix}, \ p_x = \begin{pmatrix} & -\beta & & & \\ \beta-A_2 & & & \beta-A_2 & 1 \\ & & & & \\ & \beta & & & \\ & & & & 0 \end{pmatrix}, \\ &p_z = \begin{pmatrix} \alpha+A_2 & & & \alpha-2\beta+A_2 & \\ & \beta & & & \\ & & \beta & & \\ 2\beta-\alpha-2A_2 & & & 4\beta-\alpha-2A_2 & 1 \\ & & & & 0 \end{pmatrix}, \ p_y = \begin{pmatrix} & & -\beta & & \\ & & & & \\ \beta-A_2 & & & \beta-A_2 & 1 \\ & & \beta & & \\ & & & & 0 \end{pmatrix}, \end{aligned} \quad (69)$$

where $A_1 = A^1_{tx}$, $A_2 = A^x_{tx}$, α is a free parameter and β satisfies $\beta^2 - A_2\beta + A_1 = 0$. It can be simplified by setting $\alpha = -\frac{A_2}{2}$,

$$
\begin{aligned}
p_t &= \begin{pmatrix} 2\beta - \frac{A_2}{2} & & & -\frac{A_2}{2} & 1 \\ & \beta & & & \\ & & \beta & & \\ -\frac{A_2}{2} & & & 2\beta - \frac{A_2}{2} & \\ & & & & 0 \end{pmatrix}, \ p_x = \begin{pmatrix} & & -\beta & & \\ \beta - A_2 & & & \beta - A_2 & 1 \\ & & & & \\ & & \beta & & \\ & & & & 0 \end{pmatrix}, \\
p_z &= \begin{pmatrix} \frac{A_2}{2} & & & \frac{A_2}{2} - 2\beta & \\ & \beta & & & \\ & & \beta & & \\ 2\beta - \frac{3A_2}{2} & & & 4\beta - \frac{3A_2}{2} & 1 \\ & & & & 0 \end{pmatrix}, \ p_y = \begin{pmatrix} & & -\beta & & \\ & & & & \\ \beta - A_2 & & & \beta - A_2 & 1 \\ & & \beta & & \\ & & & & 0 \end{pmatrix}.
\end{aligned} \tag{70}
$$

The other kind of representation is

$$
\begin{aligned}
p_t &= \begin{pmatrix} \gamma & & & \gamma - A_2 & 1 \\ & \lambda & & & \\ & & \lambda & & \\ 2\lambda - \gamma - A_2 & & & 2\lambda - \gamma & \\ & & & & 0 \end{pmatrix}, \ p_x = \begin{pmatrix} & & \lambda - A_2 & & \\ \lambda - A_2 & & & \lambda - A_2 & 1 \\ & & & & \\ & & A_2 - \lambda & & \\ & & & & 0 \end{pmatrix}, \\
p_y &= \begin{pmatrix} & & \lambda - A_2 & & \\ & & & & \\ \lambda - A_2 & & & \lambda - A_2 & 1 \\ & & A_2 - \lambda & & \\ & & & & 0 \end{pmatrix}, \ p_z = \begin{pmatrix} \gamma & & & \gamma - A_2 & \\ & \lambda & & & \\ & & \lambda & & \\ 2\lambda - \gamma - A_2 & & & 2\lambda - \gamma & \\ & & & & 0 \end{pmatrix},
\end{aligned} \tag{71}
$$

where γ is a free parameter and λ satisfies $\lambda^2 - A_2\lambda + A_1 = 0$. Setting $\gamma = \lambda$, it is simplified as:

$$
\begin{aligned}
p_t &= \begin{pmatrix} \lambda & & & \lambda - A_2 & 1 \\ & \lambda & & & \\ & & \lambda & & \\ \lambda - A_2 & & & \lambda & \\ & & & & 0 \end{pmatrix}, \ p_x = \begin{pmatrix} & & \lambda - A_2 & & \\ \lambda - A_2 & & & \lambda - A_2 & 1 \\ & & & & \\ & & A_2 - \lambda & & \\ & & & & 0 \end{pmatrix}, \\
p_y &= \begin{pmatrix} & & \lambda - A_2 & & \\ & & & & \\ \lambda - A_2 & & & \lambda - A_2 & 1 \\ & & A_2 - \lambda & & \\ & & & & 0 \end{pmatrix}, \ p_z = \begin{pmatrix} \lambda & & & \lambda - A_2 & \\ & \lambda & & & \\ & & \lambda & & \\ \lambda - A_2 & & & \lambda & \\ & & & & 0 \end{pmatrix}.
\end{aligned} \tag{72}
$$

It is apparent that the translation operations are entangled with t_1 and t_2 operations together in both representations.

The other subfamily is denoted by dte2b in which A^1_{tx} and A^t_{rt} are taken as independent deform parameters. Its commutation relation is

$$
\begin{aligned}
&[r_z, p_t] = A^t_{rt}(p_t - p_z), \ [r_z, p_z] = A^t_{rt}(p_t - p_z), \\
&[p_t, p_x] = A^1_{tx}t_1, \ \left[p_t, p_y\right] = A^1_{tx}t_2, \ [p_z, p_x] = A^1_{tx}t_1, \ \left[p_z, p_y\right] = A^1_{tx}t_2.
\end{aligned} \tag{73}
$$

dte2b does not have a natural representation which is a continuous deformation from the representation of Poincaré group. Moreover we can observe that the deformation group is more likely an isometry group of curved spacetime and the rotation operation does not seem compact anymore. We can ignore this kind of deformation group of E(2).

3. $A^t_{rt} + A^z_{rt} = 0$, $A^x_{tx} = 0$ and the corresponding Lie algebra is denoted by dte3 with three deform parameters, A^1_{1t}, A^t_{rt} and A^1_{tx}. The commutation relations are

$$
\begin{aligned}
&[t_1, p_t] = p_x + A^1_{1t}t_1, \ [t_1, p_z] = p_x + A^1_{1t}t_1, \ [r_z, p_t] = [r_z, p_z] = A^t_{rt}(p_t - p_z), \ [r_z, p_x] = p_y - A^1_{1t}t_2, \\
&\left[r_z, p_y\right] = -p_x - A^1_{1t}t_1, \ [p_t, p_z] = A^1_{1t}(p_t - p_z), \ [p_t, p_x] = [p_z, p_x] = A^1_{tx}t_1, \ \left[p_t, p_y\right] = \left[p_z, p_y\right] = A^1_{tx}t_2 - A^1_{1t}p_y.
\end{aligned} \tag{74}
$$

As in the case of dte2, dte3 can be specified into two subfamilies, for there is only one independent parameter from A^t_{rt} and A^1_{1t} via the linear combination between generators when $A^1_{1t} \neq 0$.

The first subfamily is denoted by det3a, in which we take A^1_{1t}, A^1_{tx} as deform parameters and the deformed commutation relations are

$$
\begin{aligned}
&[t_1, p_t] = p_x + A^1_{1t}t_1, \ [t_1, p_z] = p_x + A^1_{1t}t_1, \ [r_z, p_x] = p_y - A^1_{1t}t_2, \ \left[r_z, p_y\right] = -p_x - A^1_{1t}t_1, \\
&[p_t, p_x] = [p_z, p_x] = A^1_{tx}t_1, \ [p_t, p_z] = A^1_{1t}(p_t - p_z), \ \left[p_t, p_y\right] = \left[p_z, p_y\right] = A^1_{tx}t_2 - A^1_{1t}p_y.
\end{aligned} \tag{75}
$$

Like what is encountered in dte2, the first order of some $\overline{A}_{ij}^{k}$ in eq. (17) does not give contribution but their second order does to the perturbation expansion of its matrix representation. There are two inequivalent representations for this kind. The first one is

$$\begin{aligned}
&p_t=\begin{pmatrix} 2\alpha+\beta & & & \beta & 1\\ & \alpha & & & \\ & & \alpha & & \\ A_1-\beta & & & A_1+2\alpha-\beta & \\ & & & & 0\end{pmatrix},\ p_x=\begin{pmatrix} & -A_1-\alpha & & & \\ \alpha & & & \alpha & 1\\ & & & & \\ & A_1+\alpha & & & \\ & & & & 0\end{pmatrix},\\
&p_y=\begin{pmatrix} & & -\alpha & & \\ & & & & \\ A_1+\alpha & & & A_1+\alpha & 1\\ & & \alpha & & \\ & & & & 0\end{pmatrix},\ p_z=\begin{pmatrix} \beta-A_1 & & & \beta-A_1-2\alpha & \\ & \alpha & & & \\ & & \alpha & & \\ 2A_1-\beta+2\alpha & & & 2A_1-\beta+4\alpha & 1\\ & & & & 0\end{pmatrix},
\end{aligned}\tag{76}$$

where β is a free parameter and α satisfies $A_2+\alpha\left(A_1+\alpha\right)=0$. By taking $\beta=\frac{A_1}{2}$, it is simplified to

$$\begin{aligned}
&p_t=\begin{pmatrix} 2\alpha+\frac{A_1}{2} & & & \frac{A_1}{2} & 1\\ & \alpha & & & \\ & & \alpha & & \\ \frac{A_1}{2} & & & 2\alpha+\frac{A_1}{2} & \\ & & & & 0\end{pmatrix},\ p_x=\begin{pmatrix} & -A_1-\alpha & & & \\ \alpha & & & \alpha & 1\\ & & & & \\ & A_1+\alpha & & & \\ & & & & 0\end{pmatrix},\\
&p_y=\begin{pmatrix} & & -\alpha & & \\ & & & & \\ A_1+\alpha & & & A_1+\alpha & 1\\ & & \alpha & & \\ & & & & 0\end{pmatrix},\ p_z=\begin{pmatrix} -\frac{A_1}{2} & & & -2\alpha-\frac{A_1}{2} & \\ & \alpha & & & \\ & & \alpha & & \\ 2\alpha+\frac{3A_1}{2} & & & 4\alpha+\frac{3A_1}{2} & 1\\ & & & & 0\end{pmatrix}.
\end{aligned}\tag{77}$$

The second representation is

$$\begin{aligned}
&p_t=\begin{pmatrix} \lambda & & & A_1+\lambda & 1\\ & \gamma & & & \\ & & \gamma & & \\ A_1+2\gamma-\lambda & & & 2\gamma-\lambda & \\ & & & & 0\end{pmatrix},\ p_x=\begin{pmatrix} & \gamma & & & \\ \gamma & & & \gamma & 1\\ & & & & \\ & -\gamma & & & \\ & & & & 0\end{pmatrix},\\
&p_y=\begin{pmatrix} & & A_1+\gamma & & \\ & & & & \\ A_1+\gamma & & & A_1+\gamma & 1\\ & & -A_1-\gamma & & \\ & & & & 0\end{pmatrix},\ p_z=\begin{pmatrix} \lambda & & & A_1+\lambda & \\ & \gamma & & & \\ & & \gamma & & \\ A_1+2\gamma-\lambda & & & 2\gamma-\lambda & 1\\ & & & & 0\end{pmatrix},
\end{aligned}\tag{78}$$

where λ is a free parameter and γ satisfies $A_2+\gamma\left(A_1+\gamma\right)=0$. By taking $\lambda=\gamma$, it is simplified to

$$p_t=\begin{pmatrix} \gamma & & & A_1+\gamma & 1\\ & \gamma & & & \\ & & \gamma & & \\ A_1+\gamma & & & \gamma & \\ & & & & 0\end{pmatrix},\ p_x=\begin{pmatrix} & \gamma & & & \\ \gamma & & & \gamma & 1\\ & & & & \\ & -\gamma & & & \\ & & & & 0\end{pmatrix},\ p_y=\begin{pmatrix} & & A_1+\gamma & & \\ & & & & \\ A_1+\gamma & & & A_1+\gamma & 1\\ & & -A_1-\gamma & & \\ & & & & 0\end{pmatrix},\ p_z=\begin{pmatrix} \gamma & & & A_1+\gamma & \\ & \gamma & & & \\ & & \gamma & & \\ A_1+\gamma & & & \gamma & 1\\ & & & & 0\end{pmatrix}.\tag{79}$$

It is apparent again as in the DTE2a that the translation operations are entangled with t_1 and t_2 operations together in both representations.

The second subfamily is denoted by dte3b, in which we take A_{rt}^{t}, A_{tx}^{1} as deform parameters and the deformed commutation relations are

$$[r_z,p_t]=[r_z,p_z]=A_{rt}^{t}\left(p_t-p_z\right),\ \ [p_t,p_x]=[p_z,p_x]=A_{tx}^{1}t_1,\ \ \left[p_t,p_y\right]=\left[p_z,p_y\right]=A_{tx}^{1}t_2.\tag{80}$$

the corresponding deformed matrix representation is

$$p_t=\begin{pmatrix} & & & A_1 & 1\\ & -A_1 & & & \\ & & -A_1 & & \\ -A_1 & & & -2A_1 & \\ & & & & 0\end{pmatrix},\ r_z=\begin{pmatrix} A_2 & & & A_2 & \\ & & -1 & & \\ & 1 & & & \\ -A_2 & & & -A_2 & \\ & & & & 0\end{pmatrix},\ p_x=\begin{pmatrix} & -A_1 & & & \\ -A_1 & & & -A_1 & 1\\ & & & & \\ & A_1 & & & \\ & & & & 0\end{pmatrix},\ p_z=\begin{pmatrix} & & & A_1 & \\ & -A_1 & & & \\ & & -A_1 & & \\ -A_1 & & & -2A_1 & 1\\ & & & & 0\end{pmatrix},\tag{81}$$

where the single parameter group element representation corresponding to r_z is

$$R_z(\theta) = \begin{pmatrix} 1+A_2\theta & & & A_2\theta \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ -A_2\theta & & & 1-A_2\theta \end{pmatrix}, \tag{82}$$

a reasonable rotation operation not only in the xy plane but also in the rotated tz plane as in DTE1. The translation operations are entangled with t_1 and t_2 operations together again.

The common features of DTE are that the rotation operation is not only in the xy plane but also in the rotated tz plane and the translation operations are entangled with t_1 and t_2 operations together.

4.6 The deformation group of ISO(3)

SO(3) group has three generators r_x, r_y, r_z. The deformation of its semi-direct product with $T(4)$ has two deform parameters A^x_{tx}, A^3_{xy}, where 3 represents r_z. The second order constrain condition is

$$A^x_{tx}A^3_{xy} = 0. \tag{83}$$

The deformation group DISO(3) therefore is specified into two classes. The first class is denoted by diso(3)1, in which the deform parameter is taken as $A_1 = A^3_{xy}$ and the commutation relations are

$$\left[p_x, p_y\right] = A_1 r_z,\ \ [p_z, p_x] = A_1 r_y,\ \ \left[p_y, p_z\right] = A_1 r_x. \tag{84}$$

The natural matrix representation is

$$p_t = \begin{pmatrix} \alpha & & & & 1 \\ & \alpha & & & \\ & & \alpha & & \\ & & & \alpha & \\ & & & & 0 \end{pmatrix},\ p_x = \begin{pmatrix} & \beta & & & \\ \alpha & & & & 1 \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$
$$p_y = \begin{pmatrix} & & \beta & & \\ & 0 & & & \\ \alpha & & & & 1 \\ & & & 0 & \\ & & & & 0 \end{pmatrix},\ p_z = \begin{pmatrix} & & & \beta & \\ & 0 & & & \\ & & 0 & & \\ \alpha & & & & 1 \\ & & & & 0 \end{pmatrix}, \tag{85}$$

where α and β satisfy $\alpha\beta + A_1 = 0$. Hence there are two ways to get simplification.

By the first way, by taking $\beta = \alpha$ if $A_1 < 0$, we have

$$p_t = \begin{pmatrix} \alpha & & & & 1 \\ & \alpha & & & \\ & & \alpha & & \\ & & & \alpha & \\ & & & & 0 \end{pmatrix},\ p_x = \begin{pmatrix} & \alpha & & & \\ \alpha & & & & 1 \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$
$$p_y = \begin{pmatrix} & & \alpha & & \\ & 0 & & & \\ \alpha & & & & 1 \\ & & & 0 & \\ & & & & 0 \end{pmatrix},\ p_z = \begin{pmatrix} & & & \alpha & \\ & 0 & & & \\ & & 0 & & \\ \alpha & & & & 1 \\ & & & & 0 \end{pmatrix}, \tag{86}$$

where $\alpha^2 = -A_1$.

By the second way, by taking $\beta = -\alpha$ if $A_1 > 0$, we have

$$p_t = \begin{pmatrix} \alpha & & & & 1 \\ & \alpha & & & \\ & & \alpha & & \\ & & & \alpha & \\ & & & & 0 \end{pmatrix},\ p_x = \begin{pmatrix} & -\alpha & & & \\ \alpha & & & & 1 \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$
$$p_y = \begin{pmatrix} & & -\alpha & & \\ & 0 & & & \\ \alpha & & & & 1 \\ & & & 0 & \\ & & & & 0 \end{pmatrix},\ p_z = \begin{pmatrix} & & & -\alpha & \\ & 0 & & & \\ & & 0 & & \\ \alpha & & & & 1 \\ & & & & 0 \end{pmatrix}, \tag{87}$$

where $\alpha^2 = A_1$.

The second family of deformation, denoted by diso(3)2, consists of deformation with deform parameters $A_1 = A^x_{tx}$. The deformed commutation relations are

$$[p_t, p_i] = A^x_{tx} p_i,\ \ i = x, y, z. \tag{88}$$

There are three kinds of representation therefore. The deformed representation matrices are

1.

$$p_t = \begin{pmatrix} \alpha & & & & 1 \\ & A_1 & & & \\ & & A_1 & & \\ & & & A_1 & \\ & & & & 0 \end{pmatrix}, \tag{89}$$

2.

$$p_t = \begin{pmatrix} -A_1 & & & & 1 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},\ p_x = \begin{pmatrix} 0 & & & & \\ -A_1 & & & & 1 \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$
$$p_y = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ -A_1 & & & & 1 \\ & & & 0 & \\ & & & & 0 \end{pmatrix},\ p_z = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ -A_1 & & & & 1 \\ & & & & 0 \end{pmatrix}, \tag{90}$$

3.

$$p_t = \begin{pmatrix} 2A_1 & & & & 1 \\ & A_1 & & & \\ & & A_1 & & \\ & & & A_1 & \\ & & & & 0 \end{pmatrix},\ p_x = \begin{pmatrix} 0 & \alpha & & & \\ & 0 & & & 1 \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$
$$p_y = \begin{pmatrix} 0 & & \alpha & & \\ & 0 & & & \\ & & 0 & & 1 \\ & & & 0 & \\ & & & & 0 \end{pmatrix},\ p_z = \begin{pmatrix} 0 & & & \alpha & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \tag{91}$$

where α is a free parameter and can be taken as A_1 in all of the three cases.

4.7 The deformation of ISO(2, 1)

Let us investigate the deformation of semi-product of three generators Lorentz subgroup SO(2, 1) with T(4), DISO(2, 1), at last. The three generators of SO(2, 1) are r_x, b_y and b_z. DISO(2, 1) has two deform parameters A^t_{tx} and A^2_{ty}, where 2 represents b_y, and a second order constrain condition,

$$A^t_{tx}A^2_{ty} = 0. \tag{92}$$

Thus DISO(2, 1)1 is specified into two families.

The first family is denoted by diso(2, 1), in which the deform parameter is taken as $A_1 = A^2_{ty}$ and the deformed commutation relations are

$$\left[p_t, p_y\right] = A_1 b_y,\ \ [p_t, p_z] = A_1 b_z,\ \ \left[p_y, p_z\right] = -A_1 r_x, \tag{93}$$

as well as the representation is

$$\begin{aligned} p_x &= \begin{pmatrix} \alpha &&&& \\ & \alpha &&& 1 \\ && \alpha && \\ &&& \alpha & \\ &&&& 0 \end{pmatrix},\ p_t = \begin{pmatrix} & \alpha &&& 1 \\ \beta &&&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 &&&& \\ && -\beta && \\ & \alpha &&& 1 \\ &&& 0 & \\ &&&& 0 \end{pmatrix},\ p_z = \begin{pmatrix} 0 &&&& \\ &&& -\beta & \\ && 0 && \\ & \alpha &&& 1 \\ &&&& 0 \end{pmatrix}, \end{aligned} \tag{94}$$

where α and β satisfy $\alpha\beta + A_1 = 0$. Thus it can be simplified according to the value of A_1.

When $A_1 > 0$, we can take $\beta = -\alpha$ and get

$$\begin{aligned} p_x &= \begin{pmatrix} \alpha &&&& \\ & \alpha &&& 1 \\ && \alpha && \\ &&& \alpha & \\ &&&& 0 \end{pmatrix},\ p_t = \begin{pmatrix} & \alpha &&& 1 \\ -\alpha &&&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 &&&& \\ && \alpha && \\ & \alpha &&& 1 \\ &&& 0 & \\ &&&& 0 \end{pmatrix},\ p_z = \begin{pmatrix} 0 &&&& \\ &&& \alpha & \\ && 0 && \\ & \alpha &&& 1 \\ &&&& 0 \end{pmatrix}, \end{aligned} \tag{95}$$

where $\alpha = \pm\sqrt{A_1}$.

When $A_1 < 0$, we can take $\beta = \alpha$ and get

$$\begin{aligned} p_x &= \begin{pmatrix} \alpha &&&& \\ & \alpha &&& 1 \\ && \alpha && \\ &&& \alpha & \\ &&&& 0 \end{pmatrix},\ p_t = \begin{pmatrix} & \alpha &&& 1 \\ \alpha &&&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 &&&& \\ && -\alpha && \\ & \alpha &&& 1 \\ &&& 0 & \\ &&&& 0 \end{pmatrix},\ p_z = \begin{pmatrix} 0 &&&& \\ &&& -\alpha & \\ && 0 && \\ & \alpha &&& 1 \\ &&&& 0 \end{pmatrix}, \end{aligned} \tag{96}$$

where $\alpha = \pm\sqrt{-A_1}$.

The second family is denoted by diso(2, 1)2, in which the deform parameter is taken as $A_1 = A^t_{tx}$ and the deformed commutation relations are

$$[p_x, p_i] = -A_1 p_i,\ \ i = t, y, z, \tag{97}$$

as well as the representation is

$$\begin{aligned} p_x &= \begin{pmatrix} \alpha - A_1 &&&& \\ & \beta &&& 1 \\ && \alpha - A_1 && \\ &&& \alpha - A_1 & \\ &&&& 0 \end{pmatrix},\ p_t = \begin{pmatrix} 0 & \alpha &&& 1 \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 &&&& \\ & 0 &&& \\ & \alpha & 0 && 1 \\ &&& 0 & \\ &&&& 0 \end{pmatrix},\ p_z = \begin{pmatrix} 0 &&&& \\ & 0 &&& \\ && 0 && \\ & \alpha && 0 & 1 \\ &&&& 0 \end{pmatrix}, \end{aligned} \tag{98}$$

where α and β satisfy $\alpha(\alpha - \beta) = 0$. Thus it can be simplified in two ways.

By the first way, $\alpha = 0$ and hence only the representation of p_x is deformed,

$$p_x = \begin{pmatrix} -A_1 &&&& \\ & -A_1 &&& 1 \\ && -A_1 && \\ &&& -A_1 & \\ &&&& 0 \end{pmatrix}. \tag{99}$$

Second, we take $\alpha = \beta = A_1$, the representation is simplified as:

$$\begin{aligned} p_t &= \begin{pmatrix} 0 & A_1 &&& 1 \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix},\ p_x = \begin{pmatrix} 0 &&&& \\ & A_1 &&& 1 \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 &&&& \\ & 0 &&& \\ & A_1 & 0 && 1 \\ &&& 0 & \\ &&&& 0 \end{pmatrix},\ p_z = \begin{pmatrix} 0 &&&& \\ & 0 &&& \\ && 0 && \\ & A_1 && 0 & 1 \\ &&&& 0 \end{pmatrix}. \end{aligned} \tag{100}$$

4.8 Summary, conclusion and outlook

Now we investigate the deformation the semi-product of all of three and four generators Lorentz subgroups with T(4) and obtain their natural representations. We list the deformation classification and the brief remark on their characters and their natural representations in Table 1.

In summary, the deformation of Poincaré group itself is the de Sitter group which is the isometry of maximal symmetric space of four dimensional spacetime, i.e., the isometry group of a curved de Sitter spacetime.

The deformation of ISIM can be classified into two families. One family is DISIM, in which SIM part is undeformed. There are many equivalent deformations which are connected with each other by redefinition of generators. For some cases there are a family of equivalent natural representations. The rotation and boost operation obtain additional accompanied scale transformation in all cases. The other family, in which the SIM part is deformed, can be divided into two subfamilies. The first subfamily is XDISIM1. Similar to family DISIM, there are also many equivalent deformations which are connected with each other by redefinition of

Table 1 The deformation of semi-product Poincaré subgroups

Subgroup	Deformation family	Deformation subfamily	Natural rep.	Remark
Poincaré	de Sitter	de Sitter	1	the isometry group of maximal symmetric space of 4-spacetime
ISIM	DISIM (SIM undeformed)	DISIM	1	much equivalent deformation corresponding to generators redefinition additional accompanied dilatation for rotation and boost operation
	XDISIM1 (SIM deformed)	XDISIM1	1	much equivalent deformation corresponding to generators redefinition additional accompanied dilatation for rotation and boost operation
	XDISIM2 (SIM deformed)	XDISIM2	1	additional accompanied dilatation for rotation operation additional accompanied dilatation for boost operation
IHOM	DIHOM1 (WDISIM)	DIHOM1 (WDISIM)	1	much equivalent representations corresponding to generators redefinition additional accompanied dilatation for boost operation same structure as the corresponding part of DISIM
	DIHOM2 (DIHOM)	DIHOM2 (DIHOM)	1	no natural representations inherited from Poincaré group additional accompanied dilatation for boost operation
TE(2)	DTE1	DTE1	1	additional accompanied dilatation for rotation operation rotation operation not only in *xy* plane but also in rotated *tz* plane
	DTE2	DTE2a	2	translations are entangled with t_1 and t_2 operations
		DTE2b	0	no natural representation inherited from Poincaré group
	DTE3	DTE3a	2	translations are entangled with t_1 and t_2 operations
		DTE3b	1	translations are entangled with t_1 and t_2 operations rotation operation not only in *xy* plane but also in rotated *tz* plane
ISO(3)	DISO(3)1	DISO(3)1	1	inequivalent representation corresponding to different signs of deform parameter only translations operations deformed
	DISO(3)2	DISO(3)2	3	three inequivalent representations only translations operations deformed
ISO(2, 1)	DISO(2, 1)1	DISO(2, 1)1	1	inequivalent representation corresponding to different signs of deform parameter only translations operations deformed
	DISO(2, 1)2	DISO(2, 1)2	2	two inequivalent representations only translations operations deformed

generators. There are also a family of equivalent natural representations. Both deformed R_z and deformed B_Z obtain additional accompanied scale transformation. The second subfamily XDISIM1 also has a family of equivalent natural representations and both deformed R_z and deformed B_Z obtain additional accompanied scale transformation. The deformed rotation operation can be a meaningful rotation only if the additional accompanied scale factor is one, i.e, the corresponding deform parameter vanishes.

The deformation of IHOM with HOM part undeformed is classified into two families. The first family is DIHOM1 which is the same deformed group XDISIM1 short of one generator r_z. The natural representation is the same as XDISIM1. The other family DIHOM2 is totally different from DIHOM1. The $5-d$ representation of DIHOM2 reveals that it is not the natural representation inherited from Poincaré group. The DIHOM2 should be the symmetry group of a curved spacetime similar to de Sitter group.

The deformation of TE with E(2) part undeformed can be classified into three families. In the first family DTE1, deformed rotation R_z is not only a rotation in the *xy* plane but also a rotation in the rotated *tz* plane and obtains additional accompanied scale transformation. The second family DTE2 is further divided into two subfamilies. For the first subfamily DTE2a, there are two inequivalent natural representations in which only the translation operators are deformed and the deformed translation operators are translation entangled with t_1 and t_2. The second subfamily DTE2b does not have a natural representation. Like DTE2, the third family DTE3 has two subfamilies. Just like DTE2a, the first one DTE3a has two inequivalent natural representations in which only the translation operators are deformed and the deformed translation operators are translation entangled with t_1 and t_2. In the second subfamily DTE3b, the deformed translation operators are translation entangled with t_1 and t_2 and the deformed rotation R_z is not only a rotation in the *xy* plane but also a rotation in the rotated *tz* plane without additional accompanied scale transformation.

The deformation of ISO(3) with SO(3) part undeformed can be classified into two families. In the first family DISO(3)1, there are two inequivalent natural representations which correspond to the sign of the deform parameter and only the translation operators deform. In the second family DISO(3)2, there are three inequivalent natural representations in which still only the translation operators deform.

Very similar to the case of ISO(3), the deformation of ISO(2, 1) with SO(2, 1) part undeformed can be classified into two families. The first family DISO(2, 1)1 is similar to the case of DISO(3)1 while the difference is that DISO(2, 1)2 has two inequivalent natural representations.

With these detailed representations and deformed as well as undeformed operators' formalism, one can search the geometry whose metric function is invariant under the action of the specified semi-product POincaré subgroup and its deformed partner and then construct the field theory in spacetime related to the invariant metric function. This procedure will build up the field theory realization of Cohen-Glashow's proposal of VSR. In our subsequent work we will present the search for invariant metric function and the construction of field theory.

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