Time Dilatation with the Clock at rest in S

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1 Derivation

As an effort for trying to understand Lorentz transformations, I'm trying to use them to derive the "time dilatation" result. Consider two reference frames, S (non-primed) and S' (primed), where S' is moving with respect to S with a velocity v.
Lorentz transformations:

$$x' = x\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} - vt\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} \tag{1}$$

and

$$t' = \left(t - \frac{vx}{c^2}\right)\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} \tag{2}$$

The clock is at rest in the *unprimed* frame. If the clock is at rest in the *unprimed* frame, then the following condition should be:

$$x_1 = x_2 \tag{3}$$

Rewriting (2) with x_1 and t_1 instead of x and t results in:

$$t'_1 = \left(t_1 - \frac{vx_1}{c^2}\right)\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1}$$

Rewriting (2) with x_2 and t_2 instead of x and t results in:

$$t'_2 = \left(t_2 - \frac{vx_2}{c^2}\right)\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1}$$

The difference $t'_2 - t'_1$ can now be written as:

$$t'_2 - t'_1 = \left(t_2 - \frac{vx_2}{c^2}\right)\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} - \left(t_1 - \frac{vx_1}{c^2}\right)\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1}$$

which is equivalent with:

$$t_2' - t_1' = \left(c^2\,(t_2 - t_1) - v\,(x_2 - x_1)\right)c^{-2}\left(\sqrt{\frac{c^2 - v^2}{c^2}}\right)^{-1} \tag{4}$$

Applying (3) in (4) results in:

$$t'_2 - t'_1 = (t_2 - t_{1)}\gamma$$

with

$$\gamma\ =\ \left(\sqrt{\frac{c^2 - v^2}{c^2}}\right)^{-1}$$

The time between two events on the clock's worldline is longer for an observer moving relative to the clock (in the clock's rest frame) than the time between those same two events in a frame that is at rest relative to the clock. For example, the time between the event of the clock reading "0 seconds" and the clock reading "100 seconds" would be 100 seconds in the clock's own rest frame (the primed frame according to the convention above), but in an unprimed frame moving at $0.6c$ relative to the clock, 125 seconds would elapse between these same two events.

In this derivation we were using the convention that the unprimed frame is the clock's rest frame.

2 Reference

JesseM (to be specified).