Chapter 1

Special Relativity

The Lorentz transformation is a linear transformation between observers, with,

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = L \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{1.1}$$

Therefore,

$$\frac{\partial \bar{x}^i}{\partial x^j} = L^i{}_j. \tag{1.2}$$

Using the chain rule, we find,

$$\frac{\partial}{\partial x^j} = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial}{\partial \bar{x}^i} = L^i{}_j \frac{\partial}{\partial \bar{x}^i}. \tag{1.3}$$

The pratial derivatives therefore transform inverse to the coordinates, and therefore represent a dual coefficient vector. Hence we can write, with lower indices,

$$\partial_i = \frac{\partial}{\partial x^i}. \tag{1.4}$$

Then,

$$\begin{pmatrix} \bar{\partial}_0 \\ \bar{\partial}_1 \\ \bar{\partial}_2 \\ \bar{\partial}_3 \end{pmatrix} = L^{-1} \begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}. \tag{1.5}$$

For the invariant vector, we write,

$$\left\langle e^i \right| \partial_i. \tag{1.6}$$

The partial derivative has to be written to the right of the category vector symbol, as, in curved space-time, it might vary with the coordinates.

The propagation of light is given by Maxwell's equation.

$$\frac{1}{c^2}\partial^t\partial_t E_l - \partial^x\partial_x E_l - \partial^y\partial_y E_l - \partial^z\partial_z E_l = 0. \tag{1.7}$$

The wave equation is therefore simply the length of the partial derivative coefficient vector, together with the metric, given by,

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.8}$$

The Lorentz transformations are therefore rotations in space-time, as they are the transformations that leave the lengths of vectors unchanged.

Let's look at rotations that only mix two of the 4 indices and let's ingnore the units for now. If the two indices are both spatial, say x and y, then we have a standard 2d rotation,

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.9}$$

as

$$\bar{x}^2 + \bar{y}^2 = \tag{1.10}$$

$$x^2\cos^2\phi + y^2\sin^2\phi - xy\sin\phi\cos\phi + x^2\sin^2\phi + y^2\cos^2\phi + xy\sin\phi\cos\phi \tag{1.11}$$

$$= x^2 + y^2. \tag{1.12}$$

For the t and x plane, we find

$$\begin{pmatrix} \bar{t} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{1.13}$$

as

$$\bar{t}^2 - \bar{x}^2 = \tag{1.14}$$

$$t^2\cosh^2\phi + x^2\sinh^2\phi - tx\sinh\phi\cosh\phi - t^2\sinh^2\phi - x^2\cosh^2\phi + tx\sinh\phi\cosh\phi \tag{1.15}$$

$$= t^2 - x^2, \tag{1.16}$$

with

$$\cosh^2\phi - \sinh^2\phi = 1, \tag{1.17}$$

and

$$\cosh\phi = \frac{e^x + e^{-x}}{2}, \quad \sinh\phi = \frac{e^x - e^{-x}}{2}. \tag{1.18}$$

This formula is not invariant under a galilean transformation. Let's find a transformation that leaves this equation invariant, by generalizing the galilean transformation as

$$\bar{t} = l_{11}t + l_{12}x/v \tag{1.19}$$

$$\bar{x} = -l_{21}vt + l_{22}x \tag{1.20}$$

Partial derivatives are then given by,

$$\frac{\partial\bar{t}}{\partial t} = l_{11} \qquad \frac{\partial\bar{t}}{\partial x} = \frac{l_{12}}{v} \tag{1.21}$$

$$\frac{\partial\bar{x}}{\partial t} = -l_{21}v \qquad \frac{\partial\bar{x}}{\partial x} = l_{22} \tag{1.22}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial\bar{t}}{\partial t}\frac{\partial}{\partial\bar{t}} + \frac{\partial\bar{x}}{\partial t}\frac{\partial}{\partial\bar{x}} = l_{11}\frac{\partial}{\partial\bar{t}} - l_{21}v\frac{\partial}{\partial\bar{x}} \tag{1.23}$$

$$\frac{\partial}{\partial x} = \frac{\partial\bar{t}}{\partial x}\frac{\partial}{\partial\bar{t}} + \frac{\partial\bar{x}}{\partial x}\frac{\partial}{\partial\bar{x}} = \frac{l_{12}}{v}\frac{\partial}{\partial\bar{t}} + l_{22}\frac{\partial}{\partial\bar{x}} \tag{1.24}$$

Inserting this into eq.(**??**), leads to,

$$l^2{}_{22}E_{\bar{x}\bar{x}} + \frac{l^2{}_{12}}{v^2}E_{\bar{t}\bar{t}} + \frac{2}{v}l_{22}l_{12}E_{\bar{x}\bar{t}} \tag{1.25}$$

$$-\frac{1}{c^2}(l^2{}_{21}v^2E_{\bar{x}\bar{x}} + l^2{}_{11}E_{\bar{t}\bar{t}} - 2l_{21}l_{11}vE_{\bar{x}\bar{t}}) = 0. \tag{1.26}$$

And therefore,

$$l^2{}_{22} - \frac{v^2}{c^2}l^2{}_{21} = 1 \tag{1.27}$$

$$l^2{}_{12} - \frac{v^2}{c^2}l^2{}_{11} = -\frac{v^2}{c^2} \tag{1.28}$$

$$l_{22}l_{12} + \frac{v^2}{c^2}l_{21}l_{11} = 0 \tag{1.29}$$

If we assume,

$$l_{22} = l_{21}, \tag{1.30}$$

then, after a straighforward calculation,

$$\begin{aligned} l_{22} &= \gamma \qquad (1.31) \\ l_{12} &= -\frac{v^2}{c^2} l_{11}, \qquad (1.32) \\ l_{11} &= \gamma, \qquad (1.33) \end{aligned}$$

with

$$\gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.34}$$

For the transformation we find,

$$\begin{aligned} \bar{t} &= \gamma(t - \frac{v}{c^2} x) \qquad (1.35) \\ \bar{x} &= \gamma(x - vt). \qquad (1.36) \end{aligned}$$

If we write t and x in the same units by using c,

$$\begin{aligned} c\bar{t} &= \gamma(ct - \frac{v}{c} x) \qquad (1.37) \\ \bar{x} &= \gamma(-\frac{v}{c} ct + x). \qquad (1.38) \end{aligned}$$

It follows that

$$\gamma = \cosh \phi. \tag{1.39}$$

$$\frac{v}{c} \gamma = \sinh \phi. \tag{1.40}$$

Changing the above assumption to,

$$l_{22} = a l_{21}, \tag{1.41}$$

for a constant a, just amounts to a redefinition of v in eq.(**??**).

We see, that space and time are not treated on an equal footing, since for space we can choose the angle ϕ freely, whereas the angle between a space and the time component is given by the relative movement of the inertial systems. If space and time were treated on an equal footing, then the matrix of the Lorentz transformation would only have entries depending on ϕ, and not on v.

The above equations are dependend on c, which is taken to be the speed of light in a vacuum. The Maxwell's equation is then only Lorentz invariant, if c refers to the speed of light in a vacuum there as well. Every other wave equation, say Maxwell's equation for light travelling in a medium with $c' \neq c$, or sound waves and so on, are all not Lorentz invariant.

Here, time is not a paramater, that parameterizes the movement of one coordinate system relative to the other, but a coordinate, and hence is represented by a coordinate axis. The axis corresponding to $\bar{t}$, is given, in the coordinate system t, x, by $\bar{x} = 0$, and hence by,

$$x = vt. \tag{1.42}$$

The $\bar{x}$ axis is given by $\bar{t} = 0$, hence

$$x = \frac{c^2}{v}t. \tag{1.43}$$

Let's look at,

$$c^2\bar{t}^2 - \bar{x}^2 = \gamma^2\left(c^2t^2 + 2vxt + \frac{v^2}{c^2}x^2\right) - \gamma^2\left(x^2 + 2xvt + v^2t^2\right) \tag{1.44}$$

$$= \gamma^2\left(c^2t^2 - x^2 + 2vxt + \frac{v^2}{c^2}x^2 - 2xvt - v^2t^2\right) \tag{1.45}$$

$$= \gamma^2\left(c^2t^2(1 - \frac{v^2}{c^2}) - x^2(1 - \frac{v^2}{c^2})\right) = c^2t^2 - x^2. \tag{1.46}$$

Let's just transform the spacial part, with

$$\bar{t} = t, \tag{1.47}$$

$$\bar{x} = l_{11}x + l_{21}y, \tag{1.48}$$

$$\bar{y} = l_{12}x + l_{22}y. \tag{1.49}$$

Partial derivatives are then given by,

$$\frac{\partial\bar{x}}{\partial x} = l_{11} \qquad \frac{\partial\bar{x}}{\partial y} = l_{21} \tag{1.50}$$

$$\frac{\partial\bar{y}}{\partial x} = l_{12} \qquad \frac{\partial\bar{y}}{\partial y} = l_{22}. \tag{1.51}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial \bar{x}}{\partial x}\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial x}\frac{\partial}{\partial \bar{y}} = l_{11}\frac{\partial}{\partial \bar{x}} + l_{12}\frac{\partial}{\partial \bar{y}} && (1.52)\\
\frac{\partial}{\partial y} &= \frac{\partial \bar{x}}{\partial y}\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial y}\frac{\partial}{\partial \bar{y}} = l_{21}\frac{\partial}{\partial \bar{x}} + l_{22}\frac{\partial}{\partial \bar{y}} && (1.53)
\end{aligned}
$$

Inserting this into eq.(**??**), leads to,

$$
l^2{}_{11}E_{\bar{x}\bar{x}} + l^2{}_{12}E_{\bar{y}\bar{y}} + 2l_{11}l_{12}E_{\bar{x}\bar{y}} \tag{1.54}
$$

$$
+l^2{}_{21}E_{\bar{x}\bar{x}} + l^2{}_{22}E_{\bar{y}\bar{y}} + 2l_{21}l_{22}E_{\bar{x}\bar{y}} = 0. \tag{1.55}
$$

And therefore,

$$
\begin{aligned}
l^2{}_{11} + l^2{}_{21} &= 1, && (1.56)\\
l^2{}_{12} + l^2{}_{22} &= 1, && (1.57)\\
l_{11}l_{12} + l_{21}l_{22} &= 0. && (1.58)
\end{aligned}
$$

This is satisfied by,

$$
\begin{aligned}
l_{11} &= l_{22} = \cos\phi, && (1.59)\\
l_{12} &= -l_{21} = \sin\phi. && (1.60)
\end{aligned}
$$

The coefficients therefore just transform under a rotation in the $x - y$ plane.

1.1 Proper Time

The length of a 4-vetor is the same for all observers, since the Lorentz transformation is a rotation in space-time. Therefore, the difference of two 4-vectors is invariant as well. Let's write for a very small difference,

$$
ds = \sqrt{(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}. \tag{1.61}
$$

The length of a 4-vector can be positive, zero or negative. Due to the invariance of ds, it it is, say, positive, then it is positive for all oberservers. The same is true foer z zero 4-vector and for a negative one. If,

$$
dx = 0 \quad dy = 0 \quad dz = 0, \tag{1.62}
$$

then,

$$ds = dt. \tag{1.63}$$

That is why a 4-vector with positive length is called time like. Let's rename ds for timelike vectors as $d\tau$ and call it the proper time.

If we use $d\tau$ to paramterize the curve of a particle in space-time, then we can define invariant four velocities and four accelerations, by,

$$v = \frac{\mathrm{d}x}{\mathrm{d}\tau}, \tag{1.64}$$

$$a = \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}. \tag{1.65}$$

Both these quantities are 4-vectors, as the numerator is given by the difference of two 4-vectors and hence transforms like a 4-vector and as the denominator is invariant.

If we parametrize the movement of a particle with the proper time, then, due to eq.(**??**) and eq.(**??**), it is the time as measured by a clock that moves together with the particle.

Measurement of distance can be reduced to time measurement, hence special relativity is about how you measure time.

$$\bar{t}^2 - t^2 = \bar{x}^2. \tag{1.66}$$