THREE TRIPLE INTEGRALS

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The desirability of investigating the triple integrals

$$I_1 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{du\,dv\,dw}{1-\cos u\cos v\cos w},$$

$$I_2 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{du\,dv\,dw}{3-\cos v\cos w-\cos w\cos u-\cos u\cos v},$$

$$I_3 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{du\,dv\,dw}{3-\cos u-\cos v-\cos w}$$

has arisen as a consequence of their having appeared in a recent paper* in ferromagnetic anisotropy by F. van Peype, a pupil of H. A. Kramers. The problem of evaluating them was proposed by Kramers to R. H. Fowler who communicated it to G. H. Hardy. The problem then became common knowledge first in Cambridge and subsequently in Oxford, whence it made the journey to Birmingham without difficulty.

It is possible to express all three of the integrals in terms of simple surds, the number π, and certain complete elliptic integrals. The elliptic integrals which occur in I_1 and I_2 are easily expressible in terms of gamma functions whose arguments are simple fractions. This is not the case, so far as I know, with the elliptic integral occurring in I_3; though this elliptic integral is quite familiar, since its modulus is one of the simplest of the set of singular moduli.

The value of I_1 appears to be fairly well known, but, for the sake of completeness, I give two methods of determining it. So far as I know, no method of evaluating I_2 and I_3 has hitherto been discovered; my methods involve, in the one case, the use of a certain expansion for the square of a complete elliptic integral and, in the other case, the use of a formula discovered by Bailey connecting the product

* *Physica*, 5 (1938), 465.

of two hypergeometric functions with a hypergeometric function of two variables of Appell's fourth type.

The results which I have obtained are as follows:

$$I_1 = \frac{4K_0^2}{\pi^2} = \frac{\Gamma^4(\frac{1}{4})}{4\pi^3},$$

$$I_2 = \frac{K_1^2\sqrt{3}}{\pi^2} = \frac{3\Gamma^6(\frac{1}{3})}{2^{14/3}\pi^4},$$

$$I_3 = (18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})(2K_2/\pi)^2,$$

where K_0, K_1, K_2 denote the complete elliptic integrals with moduli

$$\sin 45^\circ, \qquad \sin 15^\circ, \qquad (2-\sqrt{3})(\sqrt{3}-\sqrt{2})$$

respectively. From these results it is easy to compute the numerical values of I_1, I_2, and I_3, namely

$$I_1 = 1{\cdot}39320\ \ 39297,$$

$$I_2 = 0{\cdot}44822\ \ 03944,$$

$$I_3 = 0{\cdot}50546\ \ 20197.$$

I start with all three integrals in the same way, by writing

$$\tan\tfrac{1}{2}u = x, \qquad \tan\tfrac{1}{2}v = y, \qquad \tan\tfrac{1}{2}w = z,$$

and then, regarding x, y, z as Cartesian coordinates, I change to polar coordinates by the transformation

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta;$$

after writing $2\phi = \psi$, I next bisect the ψ-range of integration; the integrands at corresponding points in the two parts of the range being equal, this procedure gives

$$\int\limits_0^{\frac{1}{2}\pi} (\ldots)\,d\phi = \tfrac{1}{2}\int\limits_0^{\pi} (\ldots)\,d\psi = \int\limits_0^{\frac{1}{2}\pi} (\ldots)\,d\psi.$$

There is no difficulty in justifying the various changes which are made in the order of carrying out the integrations, because all the integrands are positive throughout and the triple integrals are transformed into integrals which are evidently convergent.

By treating I_1 in the manner just described, we get

$$I_1 = \frac{8}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(1+x^2)(1+y^2)(1+z^2)-(1-x^2)(1-y^2)(1-z^2)}$$

$$= \frac{4}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{x^2+y^2+z^2+x^2y^2z^2}$$

$$= \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\phi}{1+r^4\sin^4\theta\cos^2\theta\sin^2\phi\cos^2\phi}$$

$$= \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\psi}{1+\frac{1}{4}r^4\sin^4\theta\cos^2\theta\sin^2\psi}.$$

In the innermost integral replace the variable r by a new variable t defined by the formula

$$t = r\sin\theta\sqrt{(\tfrac{1}{2}\cos\theta\sin\psi)};$$

this transformation gives

$$I_1 = \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{dtd\theta d\psi}{(1+t^4)\sqrt{(\frac{1}{2}\cos\theta\sin\psi)}}$$

$$= \frac{4\sqrt{2}}{\pi^3}\int_0^\infty \frac{dt}{1+t^4}\cdot\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{\cos\theta}}\int_0^{\frac{1}{2}\pi}\frac{d\psi}{\sqrt{\sin\psi}}.$$

Now it is an easy exercise in contour integration to prove that

$$\int_0^\infty \frac{dt}{1+t^4} = \frac{\pi}{2\sqrt{2}},$$

and it follows from known properties of the first Eulerian integral that

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{\cos\theta}} = \int_0^{\frac{1}{2}\pi}\frac{d\psi}{\sqrt{\sin\psi}} = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})} = \frac{\Gamma^2(\frac{1}{4})}{2\sqrt{(2\pi)}}.$$

By combining these results we immediately get

$$I_1 = \frac{\Gamma^4(\frac{1}{4})}{4\pi^3},$$

which gives one of the values stated for I_1.

The second method of dealing with I_1 is to effect one of the integrations forthwith and get

$$I_1 = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \frac{dvdw}{\sqrt{(1-\cos^2 v \cos^2 w)}}.$$

We now write

$$a_0 = 1, \qquad a_n = \frac{1.3.5...(2n-1)}{2.4.6...(2n)} \quad (n \geqslant 1)$$

for brevity, expand the integrand in ascending powers of $\cos^2 v \cos^2 w$ and integrate term by term; we thus get

$$I_1 = \frac{1}{\pi^2} \sum_{n=0}^{\infty} a_n \int_0^{\pi} \int_0^{\pi} \cos^{2n} v \cos^{2n} w \, dvdw = \sum_{n=0}^{\infty} a_n^3,$$

the term-by-term integration being justified by the convergence of the series on the right.

Now it is known* that, when $0 \leqslant k \leqslant \sin 45°$,

$$K^2 = \tfrac{1}{4}\pi^2 \sum_{n=0}^{\infty} a_n^3 (2kk')^{2n};$$

and so, taking $k = \sin 45°$, we find that

$$I_1 = \frac{4K_0^2}{\pi^2},$$

which gives the other value stated for I_1.

The numerical value of I_1 is now immediately derivable from results contained in Legendre's tables of elliptic integrals and gamma-functions.

We now turn to I_2. When we transform it in the prescribed manner, we get

$$I_2 = \frac{8}{\pi^3} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{dxdydz}{3 \prod (1+x^2) - \sum (1-y^2)(1-z^2)(1+x^2)}$$

$$= \frac{2}{\pi^3} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{dxdydz}{x^2+y^2+z^2+y^2z^2+z^2x^2+x^2y^2}$$

$$= \frac{2}{\pi^3} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\infty} \frac{\sin\theta \, drd\theta d\phi}{1+r^2 \sin^2\theta (\cos^2\theta + \sin^2\theta \sin^2\phi \cos^2\phi)}$$

* Expansions of this kind have been discussed by various writers; I dealt with them myself many years ago at some length, *Quart. J. of Math.* 39 (1908), 27–51.

$$= \frac{2}{\pi^3} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\infty} \frac{\sin\theta \, dr d\theta d\psi}{1+r^2 \sin^2\theta(\cos^2\theta + \frac{1}{4}\sin^2\theta \sin^2\psi)}$$

$$= \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{d\theta d\psi}{\sqrt{(\cos^2\theta + \frac{1}{4}\sin^2\theta \sin^2\psi)}}$$

$$= \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\infty} \frac{dt d\psi}{\sqrt{\{(1+t^2)(1+\frac{1}{4}t^2 \sin^2\psi)\}}},$$

where, at the last step, $\tan\theta$ has been replaced by t.

We have now to introduce complete elliptic integrals of the first kind in which it is convenient to make explicit mention of the modulus. We therefore write $K(k)$ to denote the complete elliptic integral of the first kind with modulus k and we write $K'(k)$ to denote the complete elliptic integral of the first kind in which the modulus k' is the modulus complementary to k, so that $K'(k) = K(k')$.

Since
$$u = \int_0^{\mathrm{sc}(u,k)} \frac{dt}{\sqrt{\{(1+t^2)(1+k'^2t^2)\}}},$$

we have
$$\int_0^{\infty} \frac{dt}{\sqrt{\{(1+t^2)(1+k'^2t^2)\}}} = K(k) = K'(k'),$$

whence we obtain the preliminary result

$$I_2 = \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} K'(\tfrac{1}{2}\sin\psi)\, d\psi.$$

Now, with the meaning already assigned to a_n, we have

$$K'(k) = \sum_{n=0}^{\infty} a_n^2 \{\psi(n+1) - \psi(n+\tfrac{1}{2}) - \log k\} k^{2n};$$

and, for values of k between 0 and 1, every term in the series on the right is positive. We consequently have

$$I_2 = \frac{1}{\pi^2} \sum_{n=0}^{\infty} a_n^2 \int_0^{\frac{1}{2}\pi} \{\psi(n+1) - \psi(n+\tfrac{1}{2}) - \log(\tfrac{1}{2}\sin\psi)\} (\tfrac{1}{2}\sin\psi)^{2n}\, d\psi,$$

term-by-term integration being permissible because the series on the right proves to be convergent. To evaluate the general term on the right, we observe that

$$a_n^2\{\psi(n+1)-\psi(n+\tfrac{1}{2})-\log k\}k^{2n} = -\frac{a_n}{\sqrt{\pi}}\left[\frac{d}{d\epsilon}\left(\frac{\Gamma(n+\frac{1}{2}+\epsilon)}{\Gamma(n+1+\epsilon)}k^{2n+\epsilon}\right)\right]_{\epsilon=0},$$

and so we have

$$a_n^2\int\limits_0^{\frac{1}{2}\pi}\{\psi(n+1)-\psi(n+\tfrac{1}{2})-\log(\tfrac{1}{2}\sin\psi)\}(\tfrac{1}{2}\sin\psi)^{2n}\,d\psi$$

$$= -\frac{a_n}{\sqrt{\pi}}\left[\frac{d}{d\epsilon}\left(\frac{\Gamma(n+\frac{1}{2}+\epsilon)}{\Gamma(n+1+\epsilon)}\int\limits_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\sin\psi)^{2n+\epsilon}\,d\psi\right)\right]_{\epsilon=0}$$

$$= -\frac{a_n}{\sqrt{\pi}}\left[\frac{d}{d\epsilon}\left(\frac{\Gamma(n+\frac{1}{2}+\epsilon)\Gamma(n+\frac{1}{2}+\frac{1}{2}\epsilon)\sqrt{\pi}}{2^{2n+\epsilon+1}\Gamma(n+1+\epsilon)\Gamma(n+1+\frac{1}{2}\epsilon)}\right)\right]_{\epsilon=0}$$

$$= \tfrac{1}{4}\pi a_n^3\{3\psi(n+1)-3\psi(n+\tfrac{1}{2})+2\log 2\}(\tfrac{1}{2})^{2n},$$

the interchange of the order of differentiation and integration presenting no particular theoretical difficulty. We have consequently arrived at the result

$$I_2 = \frac{1}{4\pi}\sum_{n=0}^{\infty} a_n^3\{3\psi(n+1)-3\psi(n+\tfrac{1}{2})+2\log 2\}(\tfrac{1}{2})^{2n}.$$

Now in my previously cited paper it is proved that, for

$$0 \leqslant k \leqslant \sin 45^\circ$$

(and by symmetry also for $\sin 45^\circ \leqslant k \leqslant 1$), we have

$$K(k)K(k') = \tfrac{1}{4}\pi\sum_{n=0}^{\infty} a_n^3\{3\psi(n+1)-3\psi(n+\tfrac{1}{2})-2\log(2kk')\}(2kk')^{2n}.$$

When we compare this expansion with the formula just obtained for I_2, it is clear that

$$I_2 = \frac{K(k_1)K(k_1')}{\pi^2},$$

where k_1 is the modulus between 0 and $\sin 45^\circ$ which satisfies the condition $2k_1k_1' = \frac{1}{2}$, so that $k_1 = \sin 15^\circ$ and $k_1' = \sin 75^\circ$. If we now use Legendre's well-known result that $K(\sin 75^\circ) = K(\sin 15^\circ)\sqrt{3}$ together with his formula expressing $K(\sin 15^\circ)$ in terms of gamma-functions, we immediately find that

$$I_2 = \frac{K^2(\sin 15^\circ)\sqrt{3}}{\pi^2} = \frac{3\Gamma^6(\frac{1}{3})}{2^{14/3}\pi^4};$$

these are the formulae for I_2 stated at the beginning of this paper, and the computation of the numerical value of I_2 is merely a matter of employing a table of gamma-functions or of complete elliptic integrals.

We finally consider I_3. When we transform it in the prescribed manner, we get

$$I_3 = \frac{4}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{\sum x^2+2\sum y^2z^2+3x^2y^2z^2}$$

$$= \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\phi}{1+2r^2\sin^2\theta(\cos^2\theta+\sin^2\theta\sin^2\phi\cos^2\phi)+ +3r^4\sin^4\theta\cos^2\theta\sin^2\phi\cos^2\phi}$$

$$= \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\psi}{1+2r^2\sin^2\theta(\cos^2\theta+\frac{1}{4}\sin^2\theta\sin^2\psi)+\frac{3}{4}r^4\sin^4\theta\cos^2\theta\sin^2\psi}.$$

Now replace r by a new variable t defined by the equation $r\sin\theta = t\surd 2$ and then perform the integration with respect to θ. This procedure gives

$$I_3 = \frac{4\surd 2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{dtd\theta d\psi}{1+t^2(4\cos^2\theta+\sin^2\theta\sin^2\psi)+3t^4\cos^2\theta\sin^2\psi}$$

$$= \frac{2\surd 2}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{dtd\psi}{\sqrt{\{(1+4t^2+3t^4\sin^2\psi)(1+t^2\sin^2\psi)\}}}.$$

Next take $\tan\psi$ as a new variable ξ, and then write $\xi = \zeta/\sqrt{(1+t^2)}$ and perform the integration with respect to t. We thus get

$$I_3 = \frac{2\surd 2}{\pi^2}\int_0^\infty\int_0^\infty \frac{d\xi dt}{\sqrt{\{(1+\xi^2+4t^2+4t^2\xi^2+3t^4\xi^2)(1+\xi^2+t^2\xi^2)\}}}$$

$$= \frac{2\surd 2}{\pi^2}\int_0^\infty\int_0^\infty \frac{d\zeta dt}{\sqrt{\{(1+t^2)(1+4t^2+\zeta^2+3t^2\zeta^2)(1+\zeta^2)\}}}$$

$$= \frac{2\surd 2}{\pi^2}\int_0^\infty K'\left(\sqrt{\frac{1+\zeta^2}{4+3\zeta^2}}\right)\frac{d\zeta}{\sqrt{\{(1+\zeta^2)(4+3\zeta^2)\}}}$$

$$= \frac{2\surd 2}{\pi^2}\int_0^{\frac{1}{2}\pi} K'\left(\frac{1}{\sqrt{(4-\sin^2\chi)}}\right)\frac{d\chi}{\sqrt{(4-\sin^2\chi)}},$$

by writing $\zeta = \tan\chi$.

We now replace $K'(k)$ by a differential coefficient, as in the corresponding discussion of I_2. The expression which it is now convenient to take is not the one used in dealing with I_2 but

$$2\pi K'(k) = -\left[\frac{d}{d\epsilon}\sum_{n=0}^{\infty}\frac{\Gamma^2(n+\frac{1}{2}+\epsilon)k^{2n+2\epsilon}}{n!\,\Gamma(n+1+2\epsilon)}\right]_{\epsilon=0}.$$

The changes of order of integration, differentiation, and summation are even more easy to justify than in the case of I_2, since the range of values of k now under consideration extends only from $\frac{1}{2}$ to $1/\surd 3$ and it has not now got 0 or 1 for an end-point. We thus find that

$$I_3 = -\frac{\surd 2}{\pi^3}\left[\frac{d}{d\epsilon}\int\limits_0^{\frac{1}{2}\pi}\left(\sum_{n=0}^{\infty}\frac{\Gamma^2(n+\frac{1}{2}+\epsilon)}{n!\,\Gamma(n+1+2\epsilon)}\frac{1}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}}\right)d\chi\right]_{\epsilon=0}$$

$$= -\frac{\surd 2}{\pi^3}\left[\frac{d}{d\epsilon}\sum_{n=0}^{\infty}\left(\frac{\Gamma^2(n+\frac{1}{2}+\epsilon)}{n!\,\Gamma(n+1+2\epsilon)}\int\limits_0^{\frac{1}{2}\pi}\frac{d\chi}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}}\right)\right]_{\epsilon=0}.$$

The next step is the integration of the various powers of $4-\sin^2\chi$. We define c to be the smaller root of the quadratic equation

$$c^2+1 = 14c,$$

so that $c = 7-4\surd 3$, and we then have

$$\begin{aligned} 4c(4-\sin^2\chi) &= 2c(7+\cos 2\chi) \\ &= 1+2c\cos 2\chi+c^2 \\ &= (1+ce^{2i\chi})(1+ce^{-2i\chi}), \end{aligned}$$

whence it follows that

$$\frac{1}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}} = \frac{(4c)^{n+\frac{1}{2}+\epsilon}}{\{(1+ce^{2i\chi})(1+ce^{-2i\chi})\}^{n+\frac{1}{2}+\epsilon}}.$$

We now expand each factor in the denominator on the right in a series of ascending powers of c, multiply the two series together, and integrate. Since the integrals of the products of unequal powers of $e^{2i\chi}$ and $e^{-2i\chi}$ are all zero, this procedure gives immediately

$$\int\limits_0^{\frac{1}{2}\pi}\frac{d\chi}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}} = \tfrac{1}{2}\pi(4c)^{n+\frac{1}{2}+\epsilon}\,{}_2F_1(n+\tfrac{1}{2}+\epsilon, n+\tfrac{1}{2}+\epsilon; 1; c^2),$$

whence it follows that

$$\sum_{n=0}^{\infty}\left(\frac{\Gamma^2(n+\frac{1}{2}+\epsilon)}{n!\,\Gamma(n+1+2\epsilon)}\int_0^{\frac{1}{2}\pi}\frac{d\chi}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}}\right)$$

$$= \tfrac{1}{2}\pi(4c)^{\frac{1}{2}+\epsilon}\frac{\Gamma^2(\frac{1}{2}+\epsilon)}{\Gamma(1+2\epsilon)}\mathfrak{F}_4(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon,1;4c,c^2),$$

where $\mathfrak{F}_4$ denotes the fourth kind of Appell's hypergeometric function of two variables.

We now avail ourselves of Bailey's result* that

$$\mathfrak{F}_4(\alpha,\beta;\gamma,\alpha+\beta-\gamma+1;a(1-b),b(1-a))$$
$$= {}_2F_1(\alpha,\beta;\gamma;a)\,{}_2F_1(\alpha,\beta;\alpha+\beta-\gamma+1;b),$$

which, for a given value of the (complex) number a whose modulus is less than unity, is valid for values of b in the domain surrounding the origin for which the condition

$$\sqrt{|a(1-b)|}+\sqrt{|b(1-a)|} < 1$$

is satisfied. When

$$a(1-b) = 4c = 28-16\sqrt{3},$$
$$b(1-a) = c^2 = 97-56\sqrt{3},$$

we have $\quad \sqrt{|a(1-b)|}+\sqrt{|b(1-a)|}-1 = 10-6\sqrt{3} < 0,$

and so the requisite condition is satisfied. Moreover, when a has any given positive value between 0 and 1, the domain of values of b surrounding the origin within which the condition is satisfied contains that portion of the real axis which extends from the origin to $1-a$. We therefore must select as the values of a and b that pair of solutions of the equations

$$a(1-b) = 28-16\sqrt{3},$$
$$b(1-a) = 97-56\sqrt{3}$$

which satisfy the condition $a+b < 1$, assuming such a pair to exist. There is, however, no difficulty in verifying that

$$a = 24\sqrt{2}-14\sqrt{6}+20\sqrt{3}-34,$$
$$b = 24\sqrt{2}-14\sqrt{6}-20\sqrt{3}+35$$

are solutions of the simultaneous equations such that

$$a+b-1 = \sqrt{4608}-\sqrt{4704} < 0.$$

* W. N. Bailey, *Quart. J. of Math.* (Oxford), 4 (1933), 305–8.

We now write

$$k_2^2 \equiv b \equiv (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2,$$
$$k_3^2 \equiv 1-a \equiv (2-\sqrt{3})^2(\sqrt{3}+\sqrt{2})^2,$$

and we immediately find that

$$I_3 = -\frac{1}{\pi^2\sqrt{2}}\left[\frac{d}{d\epsilon}\left(\frac{\Gamma^2(\frac{1}{2}+\epsilon)(k_2'k_3')^{1+2\epsilon}}{\Gamma(1+2\epsilon)}F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon;k_3'^2)\times\right.\right.$$
$$\left.\left.\times F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1;k_2^2)\right)\right]_{\epsilon=0}.$$

An application of Euler's formula to the second hypergeometric function on the right gives

$$F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1;k_2^2) = (1-k_2^2)^{-\frac{1}{2}-\epsilon}F(\tfrac{1}{2}-\epsilon,\tfrac{1}{2}+\epsilon;1;-k_2^2/k_2'^2);$$

now $F(\frac{1}{2}-\epsilon,\frac{1}{2}+\epsilon;1;-k_2^2/k_2'^2)$ is an even function of ϵ, and so its differential coefficient with respect to ϵ obviously vanishes with ϵ. Consequently, when we differentiate the product of the functions

$$F(\tfrac{1}{2}-\epsilon,\tfrac{1}{2}+\epsilon;1;-k_2^2/k_2'^2),$$
$$\frac{\Gamma^2(\frac{1}{2}+\epsilon)k_3'^{1+2\epsilon}}{\Gamma(1+2\epsilon)}F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon;k_3'^2),$$

we immediately obtain the formula

$$I_3 = -\frac{1}{\pi^2\sqrt{2}}F(\tfrac{1}{2},\tfrac{1}{2};1;-k_2^2/k_2'^2)\times$$
$$\times\left[\frac{d}{d\epsilon}\left(\frac{\Gamma^2(\frac{1}{2}+\epsilon)k_3'^{1+2\epsilon}}{\Gamma(1+2\epsilon)}F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon;k_3'^2)\right)\right]_{\epsilon=0}.$$

A second application of Euler's formula now gives

$$I_3 = -\frac{k_2'k_3'}{\pi^2\sqrt{2}}F(\tfrac{1}{2},\tfrac{1}{2};1;k_2^2)\times$$
$$\times\left[\frac{d}{d\epsilon}\left(\frac{\Gamma^2(\frac{1}{2}+\epsilon)k_3'^{2\epsilon}}{\Gamma(1+2\epsilon)}F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon;k_3'^2)\right)\right]_{\epsilon=0}$$
$$= \frac{2k_2'k_3'K_2K_3\sqrt{2}}{\pi^2},$$

where K_2 and K_3 naturally denote the complete elliptic integrals with moduli k_2 and k_3.

Now, if τ_n denotes the parameter associated with the modulus k_n, that is to say that τ_n is the quotient

$$\tau_n = iK_n'/K_n,$$

where K'_n denotes the complete elliptic integral whose modulus k'_n is complementary to k_n, it is known* that k_2 and k_3 are the moduli for which

$$\tau_2 = i\sqrt{6}, \qquad \tau_3 = i\sqrt{\tfrac{2}{3}}.$$

The quarter periods K_2 and K'_3 are consequently connected by Landen's transformation, so that we have

$$K'_3 = (1+k_2)K_2,$$

in addition to the previous relation

$$K'_3 = K_3\sqrt{\tfrac{2}{3}}.$$

By means of these results the formula for I_3 reduces to

$$I_3 = \frac{2k'_2\,k'_3(1+k_2)K_2^2\,\sqrt{3}}{\pi^2};$$

and, when we replace the moduli by their values in surd form, we immediately find that

$$I_3 = (18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})(2K_2/\pi)^2,$$

which is the value stated at the beginning of the paper. In computing the numerical value of I_3 from this result, I found it quickest to use the formula

$$2K_2/\pi = (1+2e^{-\pi\sqrt{6}}+2e^{-4\pi\sqrt{6}}+2e^{-9\pi\sqrt{6}}+\ldots)^2$$

for the preliminary computation of $2K_2/\pi = 1{\cdot}00182\ 06717\ 32$.

* See e.g. Greenhill, *Proc. London Math. Soc.* (1), 19 (1889), 351.