WHAT IF THERE WERE A DIFFERENT ZERO?

ON THE HYPOTHESES WHICH LIE AT THE BASES OF ARITHMETIC

Jonathan Cender

Dedicated to Professor David Layzer in appreciation for
Space, Time, and Motion.
And my heartfelt gratitude to Richard Dedekind, Bernhard Riemann, John Wallis, Bhaskara Achārya, and Nāgārjuna for laying the groundwork.

Abstract. Riemann's analysis of hypotheses in geometry is applied to arithmetic. A hypothesis exemplified by the placeholder zero replaces the hypothesis currently underlying the empty set and the number zero. The hypothesis is based upon a testable conception of nothingness. Examples of alternatives to the empty set and the traditional number zero based upon the hypothesis are offered.

A new numeral, a modification of one used by Wallis and Riemann, represents the alternatives and provides a means for demonstrating their effectiveness. The new idea and symbol mesh well with John A. Wheeler's and Roger Penrose's notation for infinite arrays. Other zeros (*e.g.*, placeholder, digit, ordinal, exponent) and their symbol "0" need not change.

Consequences of applying the usual rules of arithmetic to the new zeros are explored. A substitute for the Dirac Delta function appears in basic arithmetic. The new zeros are not constrained by Hankel's Theorem from extending the Real and Complex number systems. An n-real-dimensional space is defined operationally. The Calculus simplifies because discontinuities and indeterminants are fewer. Particular attention is paid to the Lorentz term as used in Special Relativity to show how the new zero may provide a different view of phenomena.

Key words and phrases. zero, potential zero, empty set, Dirac Delta Function, Complex plane, infinity, transfinites, hyperreals, Special Relativity, n-real-dimensional space, point set topology, line set topology, totality numbers, panoply numbers.

December 13, 2010. Draft version for limited circulation. Written using $\LaTeX\,2_\varepsilon$.

Contents

1. What If?

> *In mathematics the art of properly stating a question is more important than the solving of it.*
>
> *Georg Cantor*

> *I have no data yet. It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts.*
>
> *Sherlock Holmes*

> *Has All —*
> *a codicil?*
>
> *Emily Dickinson*

What if there were a different zero? What would stay the same and what would change in mathematics? Is this variant zero a number that will "serve as a means of apprehending more easily and more sharply the difference of things"[0] than the current zero?

I submit these questions in the spirit of Charles Sanders Peirce's statement regarding the importance of mathematics for discovery. He wrote, the "only aim" of

> the Conditional or Hypothetical Science of *Pure Mathematics*... is to discover not how things are, but how they might be supposed to be, if not in our universe, then in some other.[1]

My response to these questions travels four main avenues of thought. They are the hypotheses of nothing used in mathematics, the nature and meaning of zero, examples of different number[2] zeros, and the use of these new zeros in physics. The third avenue, presumably of central interest for many, may be read independently of the others.

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[0]Dedekind, Richard, *Essays on the Theory of Numbers*, p. 14, translator, W. W. Beman. They can be found at Project Gutenberg's website, `http://www.gutenberg.org/files/21016/21016-pdf.pdf` The *Essays* are also collected in *God Created the Integers: The Mathematical Breakthroughs That Changed History*, edited by Stephen W. Hawking.

[1]Quoted in Apel, Karl-Otto, *Charles S. Peirce : From Pragmatism to Pragmaticism*, pp. 119-20, University of Amherst Press, 1981. Translation of *Der Denkweg von Charles S. Peirce: Eine Einführung in den amerikanischen Pragmatismus*, 1967, 1970 by Suhrkamp Verlag, Frankfurt am Main.

[2]Only an alternative to the cardinal is offered. Other zeros, e.g., digit, placeholder, exponent, and their familiar oval symbol, remain unchanged. The ordinal zero, although it may be dispensed with, need not change as evidenced by the page number on the title page and the first footnote of this paper.

The third avenue is the heart and genesis of the paper. Beginning in Section 5, alternate number zeros accompanied by a new numeral enlivens our discovery capacity. I call them the *potential*[3] zeros. Two sets of rules for specific versions of the potential zeros are laid out: the first set for basic arithmetic in Section 6 and the second for arithmetic on the Real number line in Sections 8.1 and 8.2.

The first two avenues are an after the fact effort to orient and contextualize the new zeros in a landscape of familiar ideas. They can be read independently of, and are not necessary for, the math and physics sections. The two avenues do present the reader with an opportunity to compare and evaluate the differing meanings at the core of both new and established zeros in a fresh light and on a slightly leveled playing field. I hope an exposition of these two avenues will give some respectability to at least the idea of a different zero regardless of the merits of the particular candidates offered here.

Section 2 looks at Bernhard Riemann's work on alternative hypotheses in geometry and finds that his opening remarks apply equally well to arithmetic. Section 3 identifies a commonly experienced hypothesis of nothing exemplified by the placeholder zero. This placeholder hypothesis is testable and points at a unification of something and nothing.

Sections 4 and 7 on the nature and meaning of zero start with Richard Dedekind's reluctance to include the empty set in his axioms. Using the omission as a framework, I present substitutes arising from the placeholder hypothesis for the empty set and for the zero axiom following precedents from Gottlob Frege and Giuseppe Peano.

The last avenue is perhaps the most important. Dedekind clearly states the importance of number for physics.

> It is only through the purely logical process of building up the science of numbers. . . that we are prepared accurately to investigate our notions of space and time by bringing them into relation with this number-domain created in our mind.[4]

Might zeros arising from division offer deeper insights than a zero derived from subtraction and first used for finance and accounting? Are zero numbers with a unified view of something and nothing "in our universe [or] in some other?"

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[3]I personally preferred *possibility* zeros and will sometimes refer to them this way, but a quick survey among friends revealed a near unanimous preference for the term *potential.* Upon reflection, I came to agree.

[4]Dedekind, p. 14.

2. Riemann and the Plan of the Investigation

I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought.

Richard Dedekind

We hypostasize these laws of things *into* laws of thought.

Sidney Hook

For nothing is harder than to distinguish the real things of sense
From those doubtful versions of them that the mind readily supplies.

Lucretius

While playing around with zero, some awareness arose concerning the foundational issues involved in offering an alternative[5] to one of the axioms comprising standard arithmetic. I naturally turned to Bernhard Riemann's paper *On the Hypotheses which lie at the Bases of Geometry.*[6] This work lays out a basis for investigating little questioned assumptions about "the notion of space and the first principles of constructions in space,"[7] and was a key turning point in the development of an alternative to what was the standard geometry at that time: the time-hallowed plane geometry of Euclid.

Recognizing that hypotheses about number-nothing relationships are as basic to arithmetic as those about notions of space are to geometry, and wishing to open up mathematics beyond our time-honored standard arithmetic, and finding, after some dozens of readings, his analysis

[5]Thanks to Haim Gaifman for his email of 31 March 2008 clarifying some issues in his paper *"Non-Standard Models in a Broader Perspective"* (in the 2003 AMS collection *Nonstandard Models of Arithmetic and Set Theory*, ed., A. Enayat and R. Kossak). Therefore the term alternative is used instead of non-standard to avoid confusion with existing non-standard arithmetics. They all accept the basic arithmetic axioms, including the zero axiom, as given, and make their axiom changes intentionally as well. Also at `http://www.columbia.edu/~hg17/nonstandard-02-16-04-cls.pdf`

[6]Riemann. Page references will be drawn from `http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Geom/WKCGeom.html` The translation is by William Kingdom Clifford and appeared in *Nature*, Vol. VIII (1873), Nos. 183, 184, pp. 14-17, 36, 37. The paper is also at `http://www.emis.de/classics/Riemann/` in the original German as well as English. A more recent (1970s) translation can be found in *Differential Geometry, Vol. II*, by Michael Spivak.(see footnote 17)

[7]Riemann, sentence one.

pertaining to hypotheses to be a model of brevity, insight, and clarity; and upon realizing I cannot do nearly as well, I have, therefore, simply freely substituted terms appropriate for arithmetic in a few of his sentences to convey some essential ideas.

Here are the first three sentences of *On the Hypotheses which lie at the Bases of Geometry*, as applied to arithmetic.

> It is known that arithmetic assumes as things given, a notion nothingness and its relation to the first principles of the number-concept. She gives the primitive term zero which is merely nominal, while the true determinations appear in the form of the rules of arithmetic. The relation of this notion to the number-concept remains consequently in darkness; we neither perceive whether and how far their connection is necessary; nor, *a priori*, whether it is possible.

Riemann then states that his means of investigation will be an innovation called the "triply extended magnitude" based on "general notions of magnitude."[8] His innovation provided a way to test existing hypotheses about space. Correspondingly, the potential zeros test existing hypotheses about number-nothing relations. With this in mind, I further lay out the nature and scope of the metamathematical issues wrestled with here by again substituting appropriately in Riemann's work.

> I have in the first place, therefore, set myself the task of constructing the notion of potential zeros out of general notions of nothing. It will follow from this that different number zeros are capable of different number-relations, and consequently that standard arithmetic is only one possible hypothetico-deductive system concerning the notion of nothing and its relation to the first principles of the number-concept. But hence flows as a necessary consequence that the propositions of arithmetic cannot be derived from general notions of nothingness and the number-concept, but that the properties that distinguish standard arithmetic from other conceivable arithmetics are only to be deduced from particular assumptions. Thus arises the problem, to discover the preferable views and assumptions among the possible views and assumptions from which number-nothing relations may be developed; a problem which from the nature of the case

[8]Riemann, paraphrase of sentence six.

> is not completely certain, since, as will be shown, there is more than one system of assumptions which serves to provide a basis for such relations—the most important hypothetico-deductive system for our present purpose being the standard arithmetic to the foundation of which so many have contributed. These systems are—like all hypothetico-deductive systems—not necessary, but can be judged on their consistency and fruitfulness. We may therefore investigate their consistency and fruitfulness, and, further, inquire about the justice of their extension beyond the limits of thought to the usual applications of arithmetic to science.[9]

I confess to being rather in awe of Riemann's work and am very happy to hang on to his coattails here. His rationale for introducing contrasting assumptions and investigating the systems consequent upon them has greatly improved my exposition of number-nothing relations. Clearly, the task is to search out and then formulate other particular assumptions about nothing in mathematical terms. And it is just here that a major difference arises between his work and mine.

Riemann's hypotheses concerning geometry, as different as they were from those of Euclid and Descartes, still shared a basic conceptual structure. By this I mean they shared a common understanding of all the relevant terms — space, point, flatness, number, magnitude, dimension, curvature, construction in space, passes over. Riemann simply made definite those terms through his triply extended magnitude in what turned out to be novel ways, the result being that standard Euclidean/Cartesian geometry became one of many curved geometries. In contrast, the traditional and potential zeros are not part of some spectrum of zeros; they are alternatives to each other. Although I hasten to assure, the potential zeros fulfill all the functions of the traditional zero in the best tradition of the Principle of Permanence.

A chasm of meaning separates the traditional zero and the potential zero. The chasm is due to differing meanings of nothing. Fortunately, the meaning of nothing at the base of the potential zeros is already used in mathematics. Since the placeholder zero exemplifies the notion, I refer to it as the placeholder hypothesis. Secondarily, the potential zeros will stretch usual notions of number a bit. The traditional zero already does that. The potential zeros just do it in a different way. Interestingly, Riemann implicitly made use of the placeholder hypothesis when

[9]Riemann, paraphrase of sentences six through ten.

he replaced the traditional zero in his Riemann Sphere[10] with what I will call a *precursor* zero. More on this in Section 5.1.

Before explicating the conceptual differences in the next section, I wish to emphasize again a lesson from Riemann's construction of his triply extended magnitude and his precursor zero. In both cases, space and nothing respectively, he changed the concepts from nominally assumed or invoked axioms into something definite. Riemann showed that assumptions thought to be "necessary and unalterable accompaniments to our thinking" were, in fact, not so necessary or so unalterable. His placing our notions of space on a more numerate, less qualitative basis made them more definite and open to empirical investigation. This lesson has already been drawn by someone who made great use of the space notions of Riemann when unifying space and time, Albert Einstein. In his words,

> The lack of definiteness which, from the point of view of empirical importance, adheres to the notion of time in classical mechanics was veiled by the axiomatic representation of space and time as things given independently of our senses. Such a use of notions—independent of the empirical basis, to which they owe their existence—does not necessarily damage science. One may however easily be led into the error of believing that these notions, whose origin is forgotten, are necessary and unalterable accompaniments to our thinking, and this error may constitute a serious danger to the progress of science.[11]

Einstein's "definiteness" provides a handy way to look at opposing meanings of nothing and assess their suitability for placing zero on a more numerate, less qualitative, basis. Such a basis is usually adjudged more suitable to the empirical purposes of science.

To begin to assess the opposing hypotheses and lay a basis for an alternative zero, the origins of the traditional zero are examined next, highlighting the unintended consequences to which it owes its existence as well as introducing the placeholder hypothesis. For a more precise comparison given in first order logic, turn to Section 7.3 and particularly 7.3.1. A more philosophical discussion is in Section 10.

[10]Also known as the Gauss sphere, the extended complex plane, or the complex projective line.

[11]Einstein, Albert, "The Fundamentals of Theoretical Physics," Chapter 12 in *Out of My Later Years*, Random House, 1993 edition, p. 69.

3. The origin and meaning of zero

And so, from all appearances, the discovery of zero was An Accident brought about by an attempt to make an unambiguous permanent record of a counting board operation.

Tobias Dantzig

Nothing can be born out of mere [nihilistic] nothingness. But from the "emptiness" of the Middle Doctrine, which is a kind of infinite potentiality, anything and everything may be born or produced, depending upon what causes happen to affect it. Various objects and phenomena appear to the ordinary beholder to be arising out of nothing. But what precedes them is not in fact [nihilistic] nothingness but the state of ku *or potentiality that Nāgārjuna has been describing.*

Daisaku Ikeda

Nothing is as nothing does.

Anonymous

How is nothing used in mathematics? I will sort out two different zero hypotheses in use and call attention to the origin of the switch from the nothing of the placeholder to the nothing of the number as an unintended consequence of switching the use of the zero symbol from digit to numeral. I focus particularly on the definiteness of the hypotheses as discussed in the previous section with an eye to laying out a basis for alternatives to the empty set and the zero axiom; alternatives suitable for a zero (really zeros) more like other numbers.

3.1. **A briefer history of nothing.**

Babylon is the earliest civilization known to have used a placeholder zero.[12] Scribes first used a blank space in their positional notation, and eventually began using a symbol to indicate more definitely whether a place was to be held or not. From the Middle East, the placeholder zero was taken to Greece and India where it met with two very different receptions: rejection in Greece and acceptance in India.

The placeholder zero was rejected in Greece by those who rejected ideas of infinity and nothingness. Although atomists such as Leucippus and his student Democritus asserted the existence of the void and

[12]The general background in this section is drawn primarily from Charles Seife's *Zero: The Biography of a Dangerous Idea* as well as from standard histories.

infinity as basic to explaining the universe, their ideas and others like them, including the import from Mesopotamia, were casualties in a culture war won by Aristotle and other proponents of a finite universe and what later came to be called a *horror vaccui*.[13]

India, on the other hand, turned out to be a welcoming haven for the placeholder zero. The mathematical newcomer fit right in with already accepted ideas about an interactive, fillable nothingness such as the pregnant void, the empty (sunya or shunya), and even a Philosophy of Void.[14] It also fit right in with a different number of digits—working just as well with the Indian base ten positional notation as with the base sixty of the Babylonian original. Indians felt it only right that a symbol representing absence could hold a place where something potentially could be present. No horror of the vacuum here. After centuries of service elsewhere, efforts began to be made to fit the placeholder digit in with other numbers. The project went hand in hand with the emergence of negative numbers prompted by the needs of merchants and financiers.

3.2. **The bank account theory of zero.** Amidst a culture supporting the normalcy of nothingness and the virtues of the void, mathematicians, notably Brahmagupta, developed the placeholder digit into a number of nothing named *sunya* from which the word zero is derived. An often cited example concerning bank accounts proved influential in elevating the placeholder to the status of number. It was argued that a *sunya* (empty) bank account, one with no money, has a value of zero money. The difference between zero as the value of an amount of money versus the nothing of not having a bank account justified zero's new status.

Changing the placeholder's symbol from digit to numeral came with some unintended consequences. After undergoing arithmetic operations the symbol still represented emptiness, but in a very different way from before. As a digit the symbol "0" represented the absence of the other digits. As a placeholder it represented the temporary absence of values associated with the place it occupied. This was only assumed in each case and not specified by the symbol. The symbol was a sort of shorthand for "the other digits aren't here now" and "that value is not here now". The shorthand performed its limited and very specific task admirably.

[13]Henning Genz's book *Nothingness: The Science of Empty Space* contains an excellent discussion concerning this conflict.

[14]Nāgārjuna's Philosophy of Sunyata. More on this in Section 10.

The same oval symbol that interacted well with the other digits faced different and more diverse tasks as a numeral. And under arithmetic operations it didn't interact at all with other numbers. For the purposes of addition and subtraction, this was great. When zero was multiplied, however, it acted differently from all the other numbers because none of the products that resulted were factorable. It annihilated other numbers; their existence and identity no more. While this was readily deemed acceptable, matters moved more slowly for division.

Consensus on the undefined status of division by zero was reached some centuries later after the lack of success of efforts by Brahmagupta and the much later ones of the great 12th Century C.E. mathematician Bhaskara Achārya (Bhaskara II). Their lack of success eventually led to the sophisticated circumlocutions[15] of calculus.

In arithmetic the relational notion of nothing associated with the placeholder zero transmogrified into stark nihilism—the empty even of emptiness—familiar to us today. It was empty of numbers, but only by being empty of everything and anything; all existences and all identities and even, presumably, the very notions of existence and identity themselves. Zero in its familiar oval form had become truly exceptional; its meaning in standard arithmetic far from its origins in the contingent, Indian sunya and the placeholder digit. The pregnant void was now barren.

Zero, lacking definiteness and numerateness, became the only number to represent a concept. Even with this violation of the Aristotelian rationale that had denied infinity the status of number, zero was accepted when it finally reached Europe because it was just too useful.

3.3. **The placeholder hypothesis.** Is there a nothing different than the "empty even of emptiness" implied by the existing number zero? A nothing whereby existence and identity persist? Obviously there is. A hypothesis of nothing in the sense of absence could be fillable much like the placeholder zero is, or at least be interrelated with something. The question is to what and how? One answer involves something like the atomist's idea of filling the void with everything. Let's look again at the placeholder and the bank account example and see how nothing need not be a dead end.

[15]Of particular interest here is Peirce's argument with the coherence of the "Weierstrassian way of regarding the calculus" as it is usually presented through the limit concept. See Apel, pp. 160, 173-74. Peirce's intent here primarily concerns a correct philosophical understanding and apprehension of continuity. Mine concerns the calculus as the limit concept pertains to the concept of nothing.

The bank account isn't just empty. It is empty of money. Zero designates the absence of amounts of money that can potentially be there. The label "0" has implied or understood referents. The placeholder zero also designates understood referents. It is not just empty, it is empty of known digits; digits that can potentially be there. The emptiness or absence in these examples is specific, definite and testable. Do we find tens, hundreds, thousandths or not? This analysis leads to the placeholder or absence hypothesis.

Looking afresh at these examples shows a clear basis in fact for a simple, commonly experienced interrelatedness of the presence of specified elements and their absence. Absence in this sense is the source for the hypothesis of absence that underlies the potential zero. It is commonly experienced because it is the usual notion of nothing used in everyday life. It is as simple as the infant's game of peek-a-boo or the child's hide-and-go-seek. We normally understand through context at least, that if some particular thing or things are absent, they can potentially be present. And if something is present, it can be absent. This interchangability is the placeholder or peek-a-boo hypothesis.

Absence as commonly experienced also makes possible alternatives to the empty set and the zero axiom. The following section begins to address these foundational issues. The next few sections after show that the potential zeros or numbers of absence, symbolized by a new numeral, can perform addition and subtraction equally as well as the Indian one while functioning more like the other numbers when engaged in multiplication and division.

At long last, the often rediscovered and much dismissed central insight Bhaskara Achārya recorded in his *Bijaganita*—a number divided by zero equals infinity—may be put in a mathematical form less easy to dismiss than when the insight is formulated in the form $n/0 = \infty$.[16]

And it has the added bonus of providing a strong defense against the "ghosts" Bishop Berkeley used to attack Isaac Newton's Calculus. Yes, Bishop, there *is* a "departed quantity."

[16] *Bijaganita*, meaning "Seed Counting," is a work of Algebra.

4. Dedekind and the nature and meaning of numbers

> *The derivative was first* used*; it was then* discovered*; it was then* explored *and* developed*; and it was finally* defined. (emphases in original)
>
> *Judith V. Grabiner*

> *By knowing things that exist, you can know that which does not exist. That is the void.*
>
> *Musashi*

> *You know what I like about summer days? They're just made for* doing *things... even if it's nothing.* Especially *if it's nothing.*
>
> *Calvin and Hobbes*

Bernhard Riemann wrote about the "merely nominal" and the "true or essential determinations"[17] within the foundations of geometry at the dawn of a new era in our understanding of number. After untold centuries of being used, discovered, explored and developed; our hypotheses concerning number were finally being axiomatized. Richard Dedekind, like Riemann a student of Gauss as well as an editor of some of Riemann's work, was at the forefront of those following in Euclid's footsteps. Working to corral the raw material at hand, he devised[18] all but one of the axioms at the foundation of standard arithmetic and set them forth in his essay *The Nature and Meaning of Numbers.*[19]

Both Dedekind and his friend Georg Cantor, the inventor of set theory, were ambivalent at best about zero and the empty set. And they were not alone. "Early set theorists and several contemporary metaphysicians reject the empty set."[20]

Regarding the missing axiom Dedekind said, "We intend here for certain reasons wholly to exclude the empty system which contains no element at all, although for other investigations it may be appropriate

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[17]Spivak (see fn. 6) substitutes essential for true in his more recent translation contained in *Differential Geometry, Vol. II.*

[18]It has come to my attention very recently that C. S. Peirce, even earlier than Dedekind, also devised axioms for basic arithmetic. Zero is not in his axioms either. At this time it seems his omission was due not to uncertainty or uneasiness about zero, but to some other reason.

[19]Dedekind, pp. 14-58, esp. p. 33. This is the second of two essays. (see fn. 0)

[20]Sorensen, Roy, *Nothingness*, The Stanford Encyclopedia of Philosophy (Spring 2009 Edition), Edward N. Zalta (ed.), `http://plato.stanford.edu/archives/spr2009/entries/nothingness/`

to imagine such a system."[21] Since, by definition, his system[22] contained elements, an empty set was a contradiction in terms. It was left to others to ignore this and complete what are now called the Dedekind-Peano Axioms of standard arithmetic by bringing in a separate axiom for zero, the only concept to become a number.

Pointing out the numerateness of the concepts counting, magnitude, quantity, continuity, and infinity may be helpful in grasping what is involved in changing zero's status as a concept by using the placeholder hypothesis. Dedekind's axioms capture the essentials of counting; the primordial procedure by which the concepts quantity and magnitude became numerate and definite. His Dedekind Cut made further use of counting numbers to unify discrete quantities and continuous magnitudes; making continuity more definite through an arithmetic procedure. He and Cantor both began the mathematization of infinity through definitions within set theory; Cantor later differentiated between denumerable (countable) and undenumerable infinities. By defining and differentiating the concept in numerical terms and thus making it definite, they answered Aristotle's objection to classing infinity as a number because it was only a concept. All of this is now commonplace.

Making the concept of nothing definite through numeration can be done based on the placeholder hypothesis. It is simply a matter of showing that numbers are present and can be made absent. In this way I will justify Dedekind's seemingly odd omission by giving alternatives to the empty set and the zero axiom in Section 7; alternatives which make the concept of nothing numerate and definite; alternatives that lay bare the empirical basis for our notions of nothing; alternatives based not on a set containing no elements, but on the commonly experienced interrelatedness of the presence of specified elements and their absence; an interrelatedness similar to that of the origin of the number zero: the placeholder zero.

In the preceding I have introduced terms *presence* and *absence* that are uncommon to numbers. From context it should be clear that the terms are used in their usual senses. The absence/presence hypothesis may equally be called the presence/nonpresence hypothesis or the here/not here hypothesis. It might also be called the gertrude hypothesis because there is no there there. It is decidedly different than the term "negative" which assigns a number a different there, rather than

[21]Dedekind, p. 21.

[22]For Dedekind system was equivalent to set.

a lack of there(ness). In fact a negative number would, when absent, stay negative.

Presence of numbers is normally assumed implicitly in mathematics. However, since it will be conjoined to absence, making presence explicit becomes necessary. This will be considered in some detail in Section 7 alongside the concepts of existence and identity in first order logic. For now let me say that presence will be introduced as a modifier of number in somewhat the way identity and existence are.

However, unlike with the existence/nonexistence relationship, the absence/presence transformation does not destroy identity. Identity as well as existence and other "numbery thingies" persist throughout the transformation. So for a number a to persist means that a retains any aspect of $\exists a$ together with whatever stays the same from left to right in the identity relationship $a = a$.

The persistence of identity does not occur in the existence/ nonexistence duality of Gottlob Frege. The question "to persist or not to persist" illustrates perfectly the "chasm of meaning" between the differing hypotheses of nothing referred to earlier in Section 2.

Axiomatizing the numerate alternatives will adhere to the precedent of history and follow the raw material they are meant to corral. In the next two sections the raw material, new number zeros obtained by traveling the well worn path of use, discovery, exploration and development; is set forth.

Before moving on let me introduce a topic underlying the foregoing. While notions such as counting and nothing derive in part from an empirical basis, some aspects of our understanding of these notions do not. They do not because they are difficult, if not impossible, to test directly. Meanings of nothingness especially utilize conventions derived from pure thought. The validity of these conventions is based on their effectiveness. The suppositional source for these conventions, on the other hand, is typically difficult to examine.

Untestable or difficult to test suppositions have influenced and informed so much of the scientific enterprise that the philosopher Karl Popper has introduced the terminology "metaphysical research programme"[23] in recognition of their importance. (By metaphysical, Popper means only the *untestability* (at least at first) of a programme of investigation and explanation.)

According to Popper the source of a programme has often arisen from a highly speculative philosophical system of thought. However

[23]Popper, Karl, *Quantum Theory and the Schism in Physics*, From the Postscript to The Logic of Scientific Discovery, 1956, 1982, ed., W. W. Bartley, III, p. 31.

unrealistic or strange the suppositions of such a system may seem to us, the habits of thought developed thereby have greatly influenced what historically constitutes knowledge.

Karl Popper's idea "metaphysical research programme" is highly relevant to the introduction of an alternative zero in this paper because just such a programme has greatly influenced and informed my thinking. It is not too much to say that I would never have even started this paper without having developed habits of thought therefrom.

Section 10 introduces the metaphysical research programme which made this paper possible. The programme is Nāgārjuna's Philosophy of Void mentioned earlier in Section 3.1 and in footnote 14. Two central features of his thought are relevant. One is defining nothingness through specifying what is absent. The other is preservation of at least some characteristics when something becomes nothing along with the potential for their reappearance. Both buttress arguments supporting the introduction of an innovative zero sure to be subject to sustained inquiry. And it's plain just neat stuff!

Since the interpretation of Nāgārjuna I've found most useful is from the standpoint of pragmatism, I will close this section with a view of its foundation as a bridge to the past. C. S. Peirce, the founder of pragmatism, placed great import on the cognizable. The philosopher Karl-Otto Apel makes clear that

> Peirce accepts Kantianism insofar as it entails the restriction of the validity of all concepts to possible experience, and calls this "Pragmatism." Peirce's rejection of incognizable things-in-themselves, owing to just this very critical restriction, leads him to the possibility—in fact the unavoidability—of a realist metaphysics, a metaphysics whose hypothetical postulates must all be fallible, but whose general concepts must be able to prove their objective validity "in the long run." This is because we cannot conceive the "real" to be anything other than that which is "cognizable."[24]

(Section 7.3.1 has more on the cognizable in relation to first order logic.)

I will argue that Nāgārjuna's philosophy, wherein Void is to be understood as "no (this particular) thing," is based on the cognizable. And if it is so based, it further buttresses the interrelatedness of presence and absence as source for a zero with an objectively valid axiomatic foundation. And an objectively valid axiomatic foundation would be of great benefit to an empirical approach to grasping reality.

[24] Apel, p. 11.

5. Toward an Alternative Zero

I shall proceed from the simple to the complex. But in war more than in any other subject we must begin by looking at the nature of the whole; for here more than elsewhere the part and the whole must always be thought of together.

Karl von Clausewitz

All so-called proofs of the impossibility of infinite numbers begin by attributing to the numbers in question all the properties of finite numbers, whereas the infinite numbers if they are to be thinkable in any form, must constitute quite a new type of number.

Georg Cantor

my simple art, which is but systematized common sense . . . starts upon the supposition that when you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.

Sherlock Holmes

The road to the potential zeros, alternative arithmetics, and other alternative mathematics began with my interest in dividing by zero. Initially expecting little more than a minor diversion, I instead found myself delving deeper and deeper into a variety of mathematical, philosophical, and historical aspects of the subject. After a few initial efforts of my own, a search resulted in an acquaintance with the Hyperreals as used by Abraham Robinson in his Nonstandard Analysis as well as papers by Jesper Carlström[25] and Anton Setzer.[26] Their work satisfied my curiosity for a time, but further rumination resulted in my pursuing the overall strategy of substituting a new zero for the old one. Why not create an improved and completely new zero?

A different zero and especially a different notion of nothing seemed to offer a good chance of overcoming the usual objections to division by zero. Unexpectedly, some possibility of evading the usual constraints of Henkal's Theorem on extending the Complex numbers also arose. All this while still satisfying the minimum and necessary requirements of a

[25] Carlström, Jesper: Wheels - on division by zero. *Mathematical Structures in Computer Science*, 14(2004): no. 1, 143-184 and on his personal webpage at `http://www2.math.su.se/~jesper/research/wheels/`

[26] Setzer, Anton, *Wheels*, a draft paper located at `http://www.cs.swan.ac.uk/~csetzer/articles/wheel.pdf`

zero: being the identity element, or "no change" number, of addition, and being the sum of a number and its additive inverse.

The overall strategy of creating a different zero grew out of playing around with a particular symbolic form. As it turned out, this symbolic form and the idea it represents has a long history of being set equal to zero. It is seductively intuitive and simple to manipulate. Even though it is not a number zero and does not signify a nothing hypothesis, taking a brief look at this precursor zero should be helpful in grasping why my modified version is a zero and does represent an idea of nothing.

5.1. **Wallis and Riemann's precursor zero.** John Wallis, whose *Arithmetica infinitorum* of 1655 did so much for the development of Analysis, also introduced the "loveknot" symbol for infinity. He soon set the expression $1/\infty$ equal to zero as part of his efforts to arithmetize geometry.[27] Wallis's infinity symbol represented "the earliest use of the Scholastic categorematic infinity in the field of arithmetic,"[28] or what was later called an actual infinity. His use of ∞ seemed consistent with set theory, but rather different from those most common today: increasing without bound and a quantity larger than any quantity. The advantages of Wallis's notation—its ease of use in arithmetic and calculus, but especially hope[29] that the usual objections to division by zero didn't apply to an actual infinity—intrigued me.

The "Wallis number," the reciprocal of infinity, has since been used in both arithmetic and geometry. Arithmetically, it is set equal to 0 in the software package Mathematica,[30] where ∞ is defined as a positive infinite quantity. Of course this isn't "really" equal to zero. Elsewhere on the website for Mathematica, it is made clear that $1/x$ only equals zero as the limit when x increases without bound,[31] or

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

At best, the notation for limits is clumsy compared to Wallis's notation. Perhaps we can simplify—devise a numerate, arithmetized zero

[27]Boyer, Carl B., *The History of the Calculus and Its Conceptual Development*, Dover Publications, 1959, p. 173. This is a reprint of *The Concepts of the Calculus, A Critical and Historical Discussion of the Derivative And the Integral*, Hafner Publishing, 1949.

[28]Boyer, p. 170.

[29]This hope was dashed, alas, but served as a valuable transitional motivation.

[30]From MathWorld–A Wolfram Web Resource. `http://reference.wolfram.com/mathematica/ref/Infinity.html`

[31]Weisstein, Eric W. "Infinity." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/Infinity.html` or any basic treatment of limits.

something like Wallis's reciprocal without having to resort to the limit concept.

Bernhard Riemann used $1/\infty$ in his curved geometry. In his hands, it became a substitute for zero at the "south pole" of what came to be called the Riemann Sphere.[32] Naturally, the infinity symbol was at the "north pole." These uses of ∞ were early, if not the earliest, examples of a one point compactification. Later, similar use of ∞ as a point at infinity was made in the affinely extended real numbers and in the real projective line.

The potential zero will, in a way, make possible generalizing the one point compactification of Riemann's zero substitute. This generalization will become important in Section 8.1 for straightening out issues relating to the origin on the Real Number Line and in Section 8.4 for like issues concerning the origin on the coordinate plane.

Riemann's geometric idea of an inverse for a point at infinity has been the most fruitful so far for division by zero. His geometry is widely used in physics partly because, as an inverse, its zero is well defined. Mathematica, for example, returns **ComplexInfinity**, or $\tilde{\infty}$, when a complex number is divided by zero.[33]

The idea has also had some influence in computer programming. The floating point arithmetic known as exact arithmetic formulated by P. J. Potts and Abbas Edalat[34] is an example. The aforementioned works by Setzer and Carlström owe a debt to their efforts.

5.2. **No solace in this *non-quanta*.** Neither Dedekind nor Cantor used an inverse of an actual infinity in their work on the foundations of arithmetic. Perhaps both wanted to avoid the long shadow cast by Bishop Berkeley's attack on infinitesimals. In any event, they did not attempt to apply the infinite to the small. Unlike Riemann substituting Wallis's number for zero, they did not make any effort to use it, or any other infinitesimal in set theory. Wallis did say "that $1/\infty$ represented an infinitely small quantity, or *non-quanta*,"[35] so it would qualify as an

[32]Also known variously as the Gauss sphere, the extended complex plane, or the complex projective line.

[33]Weisstein, Eric W. "Division by Zero." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/DivisionbyZero.html` For more, follow links in the article.

[34]Edalat, Abbas (with Peter John Potts), *A New Representation for Exact Real Numbers*, Electronic Notes in Theoretical Computer Science 6, (1997) and Potts, Peter, *Exact Real Arithmetic using Möbius Transformations*, Ph. D. Thesis, 1998. These and other papers on exact arithmetic are at `http://www.doc.ic.ac.uk/~ae/papers.html`

[35]Boyer, p. 170.

infinitesimal. And there seemed to be good reasons why infinitesimals were not mathematically workable.

Infinitesimals violate the Archimedean axiom[36] and the Dedekind cut. They did not even qualify as numbers until decades after Dedekind and Cantor's time when Thoralf Skolem's hyperintegers led to hyperreals. And hyperreals aren't any closer to being a zero substitute. They are still in violation of the Archimedean axiom and the Dedekind cut, but at least an arithmetic relationship between an infinite number and its reciprocal was defined. Berkeley's shadow shrank.

The problem with infinitesimals? They just aren't equal to nothing. They are always a little bit bigger or smaller. It's just as true in the case of $1/\infty$. Despite assertions[37] to the contrary, the 1 in the numerator doesn't quite get you to zero. Putting another number in the numerator won't help; x/∞ still wouldn't signify a different nothing hypothesis. So what other modification might? What else could be substituted for zero? And will that substitute still violate the Archimedean axiom and the Dedekind cut?

5.3. **The potential zero.** Frustrated by the recognition that Wallis's numeral simply fails as a number of nothing, but still desiring to divide by zero, I tinkered[38] with the symbol and finally hit upon the idea of just removing the number one from the numerator and leaving a blank space behind.[39] Instead of one part of infinity, it looked like no part of infinity, or the absence of any fractional part of a whole. This seemed like a zero; a zero that wasn't a primitive term anymore, but a departed quantity. Note here that I loosely considered ∞ to be "all the numbers."

This modification of Wallis's number still acted like a fraction and thus continued the advantages of his notation. Removing the number

[36] Archimedes attributes the idea to Eudoxus of Cnidus.

[37] Mathematica states $1/\infty = 0$. From MathWorld–A Wolfram Web Resource. `http://reference.wolfram.com/mathematica/ref/Infinity.html`

[38] I say tinkered, but a quote from Bill Rodgers, the marathon runner, comes closer: "If you want to win a race you have to go a little berserk."

[39] Simply removing the one from the precursor zero is simple. Interpreting it is not. The new zeros, if they are to be thinkable in any form, turn out to constitute quite a new type of number(to paraphrases Cantor's quote at the beginning of this section). Please enjoy the intricacies throughout the compare and contrast approach to elucidating the new type of number that follows. For better or worse this interpretation resulted from the persistent focus and absorption of someone who had entirely too much time on his hands. The stumbling efforts presented here are in aid of those who may wish to stumble along as well and perhaps take or make other paths.

one led to completely rethinking the number and to seeing that the ideas of numerator and denominator no longer applied. There wasn't even a blank space above the fraction bar as I thought at one time, so no Babylonian placeholder redux. The modified symbol, $\overline{\infty}$, started to become one of what I'm now calling potential zeros. The empty set seemed irrelevant which would have been congenial to Dedekind and Cantor. A different hypothesis based on a more everyday idea of nothing that I began to recognize as related to the placeholder zero slowly emerged. A new way of delineating the difference between nothing and something started to arise.

But as I ran through computation after computation with this modified symbol, my view altered again. I wondered, "What did the fraction bar do?"

5.3.1. *The absence bar.* Instead of dividing, the fraction bar hid things. I now call it an *absence* bar. Whatever was below the bar indicated absent or departed numbers. Whenever the absence bar appeared it indicated specified numbers were not present in ordinary dimensions of number. Zero began to seem numerate and definite as discussed in Sections 2 and 3. To be sure, it wasn't all that definite at this point. But this new symbol was a far cry from the purely conceptual and amorphous nature of the traditional zero.

The absence bar also, I realized much later, inverted the numbers under it. The significance of this will become apparent.

The symbol below the bar can be varied for convenience sake. It doesn't have to be the infinity symbol. But I've continued to use "∞" because of its simplicity and ease of use, and its historically intuitive connection to zero from Wallis to the present day.

The absence bar, or at least the concept behind it is of central importance. It is perhaps the one truly original part of this paper unlike much else which turned out to be merely reinvented.

5.3.2. *Paradise.* There's a wrinkle here, though, that neither Bhaskara Achārya, Wallis, nor anyone else seems to have anticipated. One of the ways an actual infinity differs from a potential infinity is in its completeness or totality(to use terms from Dedekind). Well, finities are complete, too. So, soon after beginning to realize the importance of specifying what is absent, I realized there isn't any reason to designate only infinite, or transfinite, sets as absent. True, ∞, or whatever symbol is below the absence bar, would, I expected, usually be defined in terms of an (actually) infinite set such as the Rationals, the Reals, or the Complex numbers. But this zero is adaptable and its universe can be defined in terms of a finite set. Mathematicians are free to wander through this 'paradise' of zeros creating as they will. Even strictly finitist constructivists could make a variety of zeros. I gradually began to use the term totality to capture this more general sense: thus the name *totality numbers*. Well, I've kept the name, but I had to clarify my thoughts about what exactly is absent even further. (see Section 5.4)

5.3.3. *Changing the rules.* Another result of extending the specifiability of ∞ to finite sets was the realization that rules for using infinite sets from existing mathematics such as NonStandard Analysis, Affinely Extended Arithmetic, and Cantor's Transfinites did not necessarily make sense for the potential zero. This awareness bolstered what I had come to see as my main task: creating a zero that operates within the normal rules of standard arithmetic and thus follows the spirit of the Principle of Permanence as much as possible. The potential zero in its final form would flow from the "true determinations" that result from its actual use. This is what led to the potential zero becoming one of many zeros within standard arithmetic.

The actual use of the potential zeros is the primary source for differences in the workings of totality numbers from mathematics involving other types of infinite numbers. Those differences are set forth in Section 6 where, as we will see, the totality numbers can be made to fit well with the core arithmetic of Dedekind. Other differences are covered in Section 8.

5.3.4. *The zero which project.* Which zero is selected from the 'paradise' available is important. Understanding the consequences attendant on the selection of elements for the universe represented by the symbol ∞ is crucial. To slightly amend von Clausewitz,[40] "...in mathematics more than in any other subject we must begin by looking at the nature of the whole; for here more than elsewhere the part and the whole must always be thought of together." Not only is it necessary to recognize differences between the traditional zero and the new zero, it is also necessary to deal with differences *among* the new zeros. Unlike before, arithmetic rules for zero can now be customized to suit the mathematical context.

Rules for other zeros may be different from basic arithmetic's "quantity zero." Section 8.1 covers an example based on the Real number line. It sets forth rules for zeros specific to point set topology. The definition for these specific zeros leads to rules that are not quite the same

[40] From epigraph at the beginning of this section.

as the basic rules for simple arithmetic. The central issue has to do with the origin point and its relation to a one point compactification.

5.3.5. *A multiplicative inverse.* The potential zero is a multiplicative inverse and like any multiplicative inverse $\overline{\infty} \times \infty = 1$.[41] The assertion here that zero times something doesn't equal zero is perhaps the most surprising aspect of this relationship. Unexpected as may be, a multiplicative inverse that is not a fraction is the key to a full featured zero. Reassuringly, any *other* type of number times zero is still zero. More will be said about ∞ in the next section.

The product equaling the number one follows along with $\frac{1}{\infty} \times \infty = 1$ as used by Wallis and Riemann. In both cases the multiplication may be thought of as taking place simultaneously among many one to one relationships. The key difference is that the numbers represented by the potential zero are inverses of their counterparts represented in ∞.

The triangular or "closed loop" relationship among these numbers seems to be of some significance. Beginning with a finite element subjected to unending recursion results in the presence of a new creation made up of an infinity of elements such as the Natural numbers. Zero emerges as this creation is made absent; the quantities departed. Then combining the two results in a unity, a finite element, and the beginning arises anew. All three seem to presuppose each other.

Picture $\overline{\infty} \times \infty = 1$ as a relationship among a point, all points, and no points. In natural language it might read "the absence of every thing together with every thing equals everything." A bit more fancifully—I can imagine the atomists Leucippus, Democritus, and Epicurus murmuring about an atom, all the atoms, and the Void, or perhaps the Void and all atoms forming a unity. "These, then, must alternate, substance and void, since neither exists to the exclusion of the other."[42]

5.3.6. *Panoply numbers, or zero exposed.* The infinity sign with the absence bar stripped off, the inverse of the potential zero, needs to be taken into account also. I'm calling ∞ a *panoply* number. Panoply numbers join potential zeros as a different type of totality number. The name seems appropriate since *pan* means "all encompassing," and as we'll see in the next section, Wheeler shows this notation can easily encompass all dimensions. I also liked the resonance with Riemann's "n-ply extended magnitude."[43]

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[41]Note that both here and at complex infinity on the Riemann Sphere, a single, basic equality does not hold, in this case $\overline{\infty} \neq 1 \div \infty$.

[42]Lucretius, *The Way Things Are*, Book I. Rolf Humphries translation of *De rerum natura*.

[43]Riemann, p. 14.

There's another way to slice these numbers. Preliminarily, we can say that potential zero numbers are numbers of absence and panoply and cardinal numbers are numbers of presence. The words existence and nonexistence might also be used. By contrast, I would say that multiplication by the traditional zero resulted in the cessation or annihilation of "presence" in a terminal sense; an end with no hope of regeneration. This implicit meaning of zero, a "true definition" in the sense of Riemann, constitutes what will be replaced by the potential zero. The presence/absence relationship is discussed in some detail in Sections 3 and 7.

5.4. **Wheeler's array.** I was heartened[44] to learn that the physicist John A. Wheeler has used the infinity symbol as proxy for the Real numbers. The numbers are placed in an array.[45] Dedekind refers to the same idea as Wheeler in the course of setting forth an arithmetic replacement for the geometric idea of magnitude that used to underly definitions of the Real numbers. He wrote "...the system $\mathbb{R}$ forms a well-arranged domain of one dimension extending to infinity on two opposite sides."[46] I can't speak for Dedekind, but I was certainly delighted to see Wheeler's conception of ∞ as an array matching this.

Wheeler's notation was devised to suit the needs of working with an n-real-dimensional space. Roger Penrose uses it to great effect.[47] Having ∞ stand for an array is a refreshing change from the more common usage of increasing without bound or from that of a quantity greater than any real number.

As Penrose points out, one of the challenges of working with an n-real-dimensional space is denoting the different "sizes" of the space in question. Meeting this challenge is where Wheeler's notation shines. Consider ∞^4, $\infty^{6\times10^{19}}$, and ∞^∞. Roger Penrose's comments make clear the desirability of the notation.

> Almost all the spaces of significance simply have **C** points in them. However, there is a vast difference in the 'sizes'

[44]I was also heartened to discover that Chuck Norris can divide by zero! See `http://www.chucknorrisfacts.com/page2.html` If he can, perhaps we mere mortals can, too!! Many more of his amazing math abilities can be found at `http://ck022.k12.sd.us/links/chucknorris.htm` OMG!!!

[45]Penrose, Roger, *The Road to Reality*, 2004, pp. 379-382. No indication that Wheeler uses an inverse or a subset of the Reals, though. Penrose cites p. 67 in Wheeler, J. A., *Neutrinos, Gravitation and Geometry: contribution to Rendiconti della Scuola Internazionale di Fisica' Enrico Fermi-XI, Corso, July 1959.* Zanichelli, Bologna. (Reprinted in 1982.)

[46]Dedekind, p. 2.

[47]Penrose, pp. 379, 380, 580-2, 897-907, 916-25.

> of these spaces, where... we think of this 'size' simply as the dimension [which] may be a natural number (e.g. 4, in the case of ordinary spacetime, or 6×10^{19}, in the case of the phase space considered [earlier]), or it could be infinity, such as with (most of) the Hilbert state-spaces that arise in quantum mechanics.[48]

A notation such as Wheeler's clearly conveys information about the 'sizes' of an n-real-dimensional space in a very compact way that is far more useful than Cantor's transfinites $\aleph_0$ and $\aleph_1$ can ever hope to be (where $2^{\aleph_0} = \aleph_1 = \mathbf{C}$).

Wheeler's use of ∞ and exponents is similar to mine. One difference is that exponents result quite naturally from arithmetic operations with totality numbers. And having the potential zero gives the added advantage of being able to maneuver through the various dimensions making this an operationally constructible n-space. I see this constructibility as one of the most important consequences of the potential zero. (See the sections on exponents starting with 6.17 and also 8.5 later on.)

Another difference from Wheeler's array concerns zero. Zero is in his array, but not in mine. I substitute $\frac{1}{\infty}$ which I call the Wallis number. This is technically and formally important, but I will reserve further comment on this important difference until Section 8.2 so that necessary arithmetic rules may first be given in the next section.

5.5. **Get to know your zero.** Should 'totalities' be considered numbers? I regard this question as legitimate and interesting. Penrose is very careful to refer to ∞ as an array and as a handy notation, but not as a number. After all it has no decimal equivalent so neither the panoply totality nor the zero are Real or complex. The panoply number is not a Cardinal since it doesn't indicate a quantity in the usual sense. The potential zero indicates "how many" so it may be considered a Cardinal, but it would be a clumsy ordinal number. Although 0 has recently gained acceptance as a Natural number, it seems awkward to class $\overline{\infty}$ as a Natural.

Given the extensive arithmetic interactions of the Reals and the totalities and especially considering how they extend the Reals and Complex numbers operationally, I will refer to them as numbers. I want to make it clear though, that this decision is mainly for convenience and may well be determined to be incorrect. If they do qualify as numbers, they must, as Cantor said about infinite numbers, "constitute quite a new type of number."[49]

[48] Penrose, p. 379.

[49] From epigraph at the beginning of this section.

6. An Alternative Arithmetic

In mathematics, it is indeed imperative to be absolutely clear that one's equations make strict and accurate sense. However, it is equally important not to be insensitive to 'things going on behind the scenes' which may ultimately lead to deeper insights. It is too easy to lose sight of such things by adhering too rigidly to what appears to be strictly logical ...

Roger Penrose

A wrongly perceived emptiness ruins a person of meager intelligence.
It is like a snake that is wrongly grasped or knowledge that is wrongly cultivated.

Nāgārjuna

No, I couldn't endure it, I couldn't endure it! Suppose, suppose there are even no doubts in all those calculations, suppose all that's been decided in this past month is clear as day, true as arithmetic. Lord! Even so I wouldn't dare! I couldn't endure it, I couldn't! What has this been all along? ...

Dostoyevsky

The main purpose of this section is to set out workable rules for arithmetic with the potential zero. The rules governing the interaction of zero and Reals are quite similar to those for standard arithmetic and obey the spirit, if not precisely the letter, of the Principle of Permanence. The independence of the zero axiom obviates, of course, any changes to arithmetic without the potential zero and thus any need to present those familiar rules.

The symbol "∞" in this section represents an array comprised of the Real numbers.[50] When under the absence bar, each Real is inverted. The potential zero is not a Real number so it is not included. Consequences of the rules for the potential zero lead to the other totality numbers: the panoply numbers. The rules also apply to any nonempty "subarray," finite or infinite, of the Reals.

The absence or presence of the elements of the array are what's important. Remember that the symbol "0" is no longer a number. It continues, however, in its traditional roles of placeholder, exponent, etc.

[50]This is incorrect; a simplification for clarity of exposition. It just makes more sense to wait and introduce the lone exception, the Wallis number, in Section 8.2.

Totality numbers comprised of complex numbers and the arithmetic involving complex and totality numbers are considered separately in Section 8.7.

A couple of cautions are in order here. One has to do with geometric interpretation and the other with the order of operations. In reference to geometric interpretation, we are used to a pretty close correspondence between arithmetic with Real numbers and arithmetic based on the point set topology known as the Real number line. That correspondence no longer holds in some significant and not inconsiderable ways because the new zero is no longer located at the origin of the number line.

Furthermore, the new zero turns out to be only one of a myriad of zero numbers in somewhat the same way as the imaginary i is only one of the imaginary numbers. None of the zero numbers are on the number line. Indeed, none have any geometric referent at all. All this was rather disconcerting at first. Long time habits were hard for me to break. Before concluding that some part of the arithmetic given here is incorrect simply because it no longer works in terms of the number line, compare it with the arithmetic of the Real number line given in Section 8.1.

Disconcerting as it was, breaking those long time habits was worthwhile – even liberating. I now see the different arithmetics as advantageous consequences of the contingency and adaptability of the totality numbers. In this case, two different sorts of mathematical objects are being dealt with. To put it somewhat crudely, basic arithmetic derives from counting and quantifiying things, while arithmetic on the Real number line arises from notions of space, spatial relations, and movement. By defining zeros in terms of their respective contexts, key differences between the two are highlighted. These differences will call, I hope, renewed attention to the importance of Dedekind's insistence, "Instead of [geometric concepts] I demand that arithmetic shall be developed out of itself."[51]

The other caution concerns order of operations. The familiar order will be amended to reflect the new numbers. As a first approximation, put totality numbers first. Details are in Section 6.25.

As careful as I've tried to be, errors may remain. A few doozies have been corrected already. Regardless of error, I hope this work shows enough to warrant further pursuit of this subject.

As a final comment before proceeding with the arithmetic, I remark happily on the surprise that abbreviations for the totality numbers,

[51] Dedekind, p. 5.

they being rather like containers for other numbers, turn out to be pot and pan. Let's get cooking. Or clanging...

6.1. **Zero as multiplicative inverse.** The potential zero $\overline{\infty}$ is the multiplicative inverse of the totality ∞ so that, as usual for a multiplicative inverse,

$$\overline{\infty} \times \infty = 1 \tag{1}$$

Unusually for a multiplicative inverse, it is not a fraction (remember "—" is not a fraction bar), although the similarities are obvious. But dissimilarities arise, too. Usually either the multiplicand or the multiplier could be divided on both sides of the equation and the equality would hold. That is not true here for the following case.

$$\overline{\infty} \neq 1 \div \infty$$

This is the one exception to the usual rules of solving an equality I have been able to find. Similarly, complex infinity on the Riemann Sphere fails at a single inequality. These inequalities are indicative of how closely related the Riemann Sphere and the new arithmetic are. As we move through the arithmetic, a vocabulary and grammar will develop suited to discussing this and related issues in more depth.

And perhaps it bears repeating here that the symbol below the bar can be varied for convenience sake. It doesn't have to be the infinity symbol. I've continued to use "∞" because of its simplicity and ease of use, and its historically intuitive connection to zero from Wallis to the present day.

6.2. **Division by zero.** Division by the potential zero is quite simple and straightforward since it is so similar to division by a fraction. The potential zero appears to perform the familiar "fraction somersault" so that the dividend can be multiplied by the inverse of the divisor. Here's an example.

$$4 \div \overline{\infty} = 4 \times \infty$$

First off, where did the absence bar go? Is it under ∞ at any point? Normally, a fraction bar along with the numerator and denominator will remain after the "somersault." In practice this step is often skipped. An example is dividing by $\frac{1}{8}$. Writing $\frac{8}{1}$ is usually skipped; from

$$4 \div \frac{1}{8} \quad \text{to} \quad 4 \times 8$$

Nothing like that is skipped with the potential zero inverse gyration. To the extent there is a somersault, it is "internal," meaning each element of the array inverts. But there is no step where the bar is below the

∞ symbol, not even implicitly. Division by zero results in a reveal; a becoming present. What is revealed, or "uncovered," is the inverse. And the inverse is whatever is revealed or uncovered. At any rate, since I have not been able to determine any useful purpose if the absence bar were there, eliminating it avoids clutter.

Even though it's division, the simple symbol shift here is more like the sign change from $+$ to $-$ for inverses in addition and subtraction. The totality number is a zero if the bar is shown and a panoply number if not. It's that simple.

What to do with four times ∞? The ∞ term, when attached to a natural number such as 4, merges to form a different kind of number—a panoply number. Like with the integer $+4$ or an imaginary $4i$, the multiplication operation becomes implicit and 4∞ is in final form, a number in its own right. This number is the absent array made present by division. The array, when present, may be considered orthogonal to 4, although for most purposes this is irrelevant. Division by zero transforms the dividend; the absent reappears. It may be read *four pan*.

This notation is very similar to one representing a directed infinity[52] in the software package Mathematica. A directed infinity results from computations on the Complex numbers and the Complex plane. However, "∞" in Mathematica designates "an unbounded quantity that is greater than every real number."[53] Needless to say, notational similarities do not extend to conceptual ones.

A geometric interpretation of panoply numbers will be offered in Section 8.4. The tangent of zero degrees is suggestive in this regard as well (see Section 8.3).

6.3. **Division by zero reprised.** The first example may have been too quick. Let's take the long way around; slow mo as it were.

[52]From MathWorld–A Wolfram Web Resource. http://functions.wolfram.com/Constants/DirectedInfinity/introductions/Symbols/02/

[53]Weisstein, Eric W. "Infinity." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Infinity.html

$$
\begin{aligned}
4 \div \overline{\infty} &= \frac{4}{\overline{\infty}} \\
&= \frac{4}{\overline{\infty}} \times \frac{\infty}{\infty} \\
&= \frac{4 \times \infty}{\overline{\infty} \times \infty} \\
&= \frac{4 \times \infty}{1} \\
&= 4 \times \infty \\
&= 4\infty
\end{aligned}
$$

The totality ∞ is treated like any other number when divided by itself. So $\infty \div \infty = 1$. More on this in Section 6.4.

In general,

$$\frac{x}{\overline{\infty}} = x\infty \tag{2}$$

[Note that all variables in this arithmetic section represent Real numbers unless otherwise specified.]

6.3.1. *Examples.* Before going on, here are a few more examples to breed familiarity.

$$
\begin{aligned}
\frac{32}{\overline{\infty}} &= 32\infty \\
\frac{-7.4}{\overline{\infty}} &= -7.4\infty \\
\frac{8a}{\overline{\infty}} &= 8a\infty \\
\frac{\sqrt{2}}{\overline{\infty}} &= \sqrt{2}\infty
\end{aligned}
$$

6.3.2. *An objection to division by zero answered.*

A. The objection. For division to work with the traditional zero, there must be some number n so that $4 \div 0 = n$. This is needed for the equality to hold. Normally both sides of an equality like this can be multiplied by a reciprocal of one of the factors.

For example,

$$8 \div 4 = 2$$
$$8 \times (\frac{1}{4} \times 4) = 2 \times 4$$
$$8 = 8$$

and the equality holds. But, assuming 0 has a reciprocal, when we try this with $4 \div 0 = n$

$$4 \times (\frac{1}{0} \times 0) = n \times 0$$

and cancel the zeros on the left, then

$$4 = n \times 0$$

However, $4 \neq n \times 0$. There is no n to multiply times 0 that will equal 4. The equality does not hold because any number times *this* zero equals zero. This is one of the ways to show that division by zero is undefined.

B. The reply. Let's divide 4 by zero when it's not an integer, but does have an inverse, and see if the equality holds.

$$4 \div \overline{\infty} = n \tag{3}$$
$$4 \times \overline{\overline{\infty}} = n \tag{4}$$
$$4 \times \infty = n \tag{5}$$
$$4 \times \infty \times \overline{\infty} = n \times \overline{\infty} \tag{6}$$
$$4 \times 1 = n \times \overline{\infty} \tag{7}$$
$$4 = n \times \overline{\infty} \tag{8}$$

Set n equal to the pan number 4∞. Then

$$4 = 4\infty \times \overline{\infty} \tag{9}$$
$$= 4 \times \infty \times \overline{\infty} \tag{10}$$
$$= 4 \times 1 \tag{11}$$
$$4 = 4 \tag{12}$$

and the equality holds. There *is* an n to multiply times $\overline{\infty}$ that will equal 4. The critical steps are 4, 5, and 6. Division by zero works because not quite every number times zero equals zero anymore. This zero is a defined divisor. And n can only be a distinct pan number. Multiplication of Real numbers and panoply numbers is closed as will become clear as we work through the rest of this section. In short, the new zero, unlike the traditional zero, behaves like other numbers.

The potential zero times any Real number is still zero as we shall see in Section 6.5. However, zero times a panoply number is not. The totality numbers behave a little like the imaginary i. A -1 pops out most conveniently when squaring i while a 1 or $+1$ pops out when these totality numbers get together. It's quite a wondrous thing.

6.4. **Division by the inverse of zero.** After division by zero, a logical question would be, "What about division of real numbers by its inverse, the panoply number?"

For example, does it make any sense to have an expression like

$$\frac{4}{\infty}\ ?$$

In Nonstandard Analysis any number x in the numerator simply reduces to 1 by cancellation. Cancellation is the simplest way to handle this. I am not completely satisfied that some sensible meaning may not work out. Further discussion of this matter will be reserved until Section 6.19. I hope that a possible use will be more apparent in the context of exponents.

For more, specifically, on $\frac{1}{\infty}$, the Wallis number, see Section 8.2.

6.5. **Multiplication by zero.** While my main focus was on dividing by zero, I paid little attention to multiplying by zero. There didn't seem to be any reason to do so. Any number multiplied by the potential zero should just disappear. After all that is what always happened in the past. And Reals can't go above the absence bar of this zero because there's no numerator. And if Wallis's number is treated like a hyperreal by allowing numbers above the absence bar then they would disappear under those rules, too. Zero times anything equals zero. It seemed simple. But perhaps I hadn't met my production quota yet for new numbers. My attention was drawn to the other main objection to dividing by zero.[54] While it's true that $4 \times 0 = 5 \times 0$, the equality doesn't continue to hold after both sides are divided by zero because $4 \neq 5$.

Historically, there seems to have been very little attention paid to having a different way of multiplying by zero. Bhaskara Achārya (Bhaskara II) recommended that the nonzero factors of multiples of zero should be considered in case of any further operations with that product. To put it another way, he thought that a number times zero equals zero, but that the number should not necessarily disappear from the product. It should be possible for the nonzero factor to persist or remain "contained" in the zero product just as occurs whenever nonzero

[54]Thanks Jenny.

numbers are multiplied. I've found only one attempt to do something like he proposed.

A few years ago Jesper Carlström suggested a way to implement Bhaskara Achārya's idea for sums as well as products in his Doctoral thesis.

> Note that the usual rule '$0x = 0$', which states that "zero-terms can be erased", is replaced by rules stating that zero-terms can be moved in certain ways in an expression. Indeed, . . . addition by a zero-term commutes with multiplication . . . if a zero-term occurs somewhere inside an expression, then it can be moved outside.[55]

Although my approach differs, I share Carlström's thinking that

> Introducing solutions to equations like $x + 1 = 0$ is not very different from introducing solutions to an equation [like] $x \times 2 = 1$, or from introducing new elements for division by zero.[56]

6.5.1. *The absence factor.* Another whole new world of number opened up once I realized that a number multiplied by the potential zero could just merge. Four times zero would look like $4\overline{\infty}$ and be read *pot four* or *four zero.* Zero emerged as an absence factor or element rather than an annihilation factor when dividing again by zero regenerated the original number: $4\overline{\infty} \div \overline{\infty} = 4$. (See Section 6.2 on panoply numbers for the rationale behind the merge, and Section 6.10 for dividing zeros by zero.)

No longer is there just a basic, simple, solitary zero. The products of multiplication by the absence factor can be called potential number zeros, potential zeros, or pot numbers. Multiplication by the basic zero and the other potential zeros results in Real numbers departing; becoming hidden or absent, not annihilated. All these new zeros with Real parts are an embarrassment of riches. Makes one a bit giddy, or perhaps giddier.

No longer is four times zero equal to five times zero. Indeed,

$$4 \times \overline{\infty} \neq 5 \times \overline{\infty}$$
$$4\overline{\infty} \neq 5\overline{\infty}$$

Each side, each zero, is unique. 4zero does not equal pot5. This promised to overcome the other objection to dividing by zero.

In general,

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[55] Carlström, p. 6.

[56] Carlström, personal email, 25 September 2008.

$$r \cdot \overline{\infty} = r\overline{\infty} \tag{13}$$

and for any distinct r, q,

$$\overline{\infty} \neq r\overline{\infty} \neq q\overline{\infty} \tag{14}$$

Note that these zero numbers all have the same Cardinality.

6.5.2. *Examples.* Here are a some examples to get used to the idea that multiplication by zero is no longer an annihilation, but an absence; a potential presence.

$$\begin{aligned} 936 \times \overline{\infty} &= 936\overline{\infty} \\ -23 \times \overline{\infty} &= -23\overline{\infty} \\ 4\pi \times \overline{\infty} &= 4\pi\overline{\infty} \\ 16a \times \overline{\infty} &= 16a\overline{\infty} \end{aligned}$$

6.6. **Dividing by zero numbers.** Division by pot zeros; elaborations on a basic theme.

$$\begin{aligned} 4 \div 2\overline{\infty} &= 2\infty \\ 28\pi \;/\; 7\overline{\infty} &= 4\pi\infty \\ -23 \div .4\overline{\infty} &= -57.5\infty \\ \frac{7a}{a\overline{\infty}} &= 7\infty \end{aligned}$$

In general,

$$r \div q\overline{\infty} = \frac{r}{q}\,\infty \tag{15}$$

These new "absence numbers" also led to a reconsideration of addition and subtraction of both pot and pan numbers. This will be addressed later.

6.7. **Multiplying by panoply numbers.** Panoply numbers, the result of multiplying a Real number and the panoply number, have already been introduced in Section 6.2. To briefly recap, the ∞ term, when attached to a Real number such as 4, designates a different kind of number: panoply numbers.

Like with an imaginary $4i$ or the integer $+4$, the multiplication operation becomes implicit, and 4∞ is in final form; a number in its own right. This number is the absent array made present by division. The array manifests orthogonal to 4. Division by zero results in a transformation on the dividend; a reappearance of the absent. It may be read four pan.

Also mentioned (in Section 5.1), this notation is very similar to one representing a directed infinity[57] in the software package Mathematica. A directed infinity results from computations on the Complex numbers and the Complex plane. However, "∞" in Mathematica designates "an unbounded quantity that is greater than every real number."[58] Needless to say, notational similarities do not extend to conceptual ones.

In general, any Real number times any panoply number is

$$x \times y\infty = xy\infty \tag{16}$$

A geometric interpretation of panoply numbers will be offered in Section 8.4. The tangent of zero degrees is suggestive of the geometric interpretation as shown in Section 8.3.

6.8. **Multiplying by zero numbers.** Multiplying by zero numbers is similar to multiplying by panoply numbers. In general,

$$x \times y\overline{\infty} = xy\overline{\infty} \tag{17}$$

The rules for multiplication with the potential zero are different than with the traditional zero. No longer is it just zero times anything equals zero—full stop. There's a lot more variety now. Happily that variety brings zero into line with the other numbers in terms of factoring products.

6.9. **Dividing zero by Real numbers.** Like before, but different. Dividing any zero number by any Real number still results in zero. It's just that the quotient is not the same zero.

$$\overline{\infty} \div x = \frac{1}{x}\overline{\infty} \tag{18}$$

In other words the divisor is the Real part of the absence number.

6.10. **Dividing zero numbers by zero numbers.** Dividing any zero number by another zero number always results in a Real number.

$$4\overline{\infty} \div \overline{\infty} = 4$$
$$4\overline{\infty} \div 2\overline{\infty} = 2$$

[57]From MathWorld–A Wolfram Web Resource. `http://functions.wolfram.com/Constants/DirectedInfinity/introductions/Symbols/02/`

[58]Weisstein, Eric W. "Infinity." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/Infinity.html`

In general,

$$q\overline{\infty} \div \overline{\infty} = q \tag{19}$$
$$q\overline{\infty} \;/\; r\overline{\infty} = q/r \tag{20}$$

6.10.1. *Indeterminate no more.* The expression $0/0$ is considered indeterminate. I trust the preceding makes clear why $\overline{\infty}/\overline{\infty}$ and division involving other zeros as in Equation 20 are not indeterminate.

6.11. **Multiplying zero numbers and panoply numbers.** Here's an example:

$$\begin{aligned} 4\infty \times 3\overline{\infty} &= (4 \cdot 3)(\infty \cdot \overline{\infty}) \\ &= (12)(1) \\ &= 12 \end{aligned}$$

In general,

$$x\overline{\infty} \times y\infty = xy \tag{21}$$

6.12. **Dividing panoply numbers by panoply numbers.** As already mentioned in Section 6.6, $\infty \div \infty = 1$. This is the same as with normal arithmetic and hyperreal arithmetic. A number divided by itself equals one.

Here are examples of division when panoply numbers are in both the numerator and denominator.

$$\begin{aligned} 4\infty \div \infty &= 4 \\ 4\infty \div 20\infty &= \frac{1}{5} \\ 22\pi\infty/2\infty &= 11\pi \\ 13\infty/2\pi\infty &= \frac{6.5}{\pi} \\ \frac{3.6\infty}{-0.4\infty} &= -9 \end{aligned}$$

The totality part reduces to one so that in general,

$$q\infty \div r\infty = \frac{q\infty}{r\infty} \tag{22}$$
$$= \frac{q}{r} \tag{23}$$

6.13. **Dividing panoply numbers by zeros and zeros by panoply numbers.** Expressions such as

$$4\infty \div \overline{\infty},\ \frac{.25\infty}{20\overline{\infty}},\ 4\overline{\infty} \div \infty, \text{ and } \frac{.25\overline{\infty}}{20\infty}$$

simplify to expressions involving totality numbers with exponents. Discussion of this starts in Section 6.17.

6.14. **Multiplying zeros by zeros and panoply numbers by pans.** Expressions such as

$$\overline{\infty} \times \overline{\infty}$$
$$\infty \times \infty$$
$$3\overline{\infty} \times 2\sqrt{2}\overline{\infty}$$
$$.25\infty \times 20\infty$$

also simplify to expressions involving totality numbers with exponents, and are explained beginning in Section 6.17.

6.15. **Identity element of addition.** The new zero is still the identity element of addition.

$$4 + \overline{\infty} = 4$$

No Real number changes when zero is added.

$$x + \overline{\infty} = x \tag{24}$$

Nor does any panoply number change. For example,

$$x\infty + \overline{\infty} = x\infty \tag{25}$$

Sometimes zero numbers may not be neglected, however. For example, zero plus zero is not zero.

$$\overline{\infty} + \overline{\infty} = 2\overline{\infty} \tag{26}$$

This is explained in more detail in the next section.

$\overline{\infty}$ is not the only identity element of addition involving nonzero numbers.

$$x + y\overline{\infty} = x \tag{27}$$

$$x\infty + y\overline{\infty} = x\infty \tag{28}$$

In short, any zero number may be neglected since they share the same Cardinality. For those familiar with the history of infinitesimals in the development of The Calculus, Section 8.6 includes a discussion of how Equations 24 and 27 could have precluded Berkeley's criticism of infinitesimals.

6.16. **Adding zero and panoply numbers.** Adding zero numbers seems to work about like you'd expect.

$$4\overline{\infty} + 3\overline{\infty} = 7\overline{\infty}$$

And so does adding panoply numbers.

$$9\infty + 5\infty = 14\infty$$

In general, for totality numbers with any Real parts q, r,

$$q\overline{\infty} + r\overline{\infty} = (q + r)\overline{\infty} \tag{29}$$

and

$$q\infty + r\infty = (q + r)\infty \tag{30}$$

when q and r are not additive inverses. If they are additive inverses the sum of the zero numbers is

$$x\overline{\infty} + (-x)\overline{\infty} = \frac{1}{\infty}\overline{\infty}^2 \tag{31}$$

See Section 6.23.2 for an explanation of this surprising result.

And the sum of pan numbers with Real parts that are additive inverses is *not*

$$x\infty + (-x)\infty = \overline{\infty} \tag{32}$$

See Section 6.23.3 about this.

6.17. **Exponents.** Multiplying pot and pan numbers with themselves results in situations where it makes sense to use exponents as shorthand for these many, simultaneous one to one relationships. The usual rules for exponents apply to totality numbers.

As mentioned in Section 5.4 on Wheeler's array, the potential zero makes possible an operational way to move about amidst an n-real-dimensional space. These next few sections on exponents lay out the arithmetic for doing this and is developed further in Section 8.5.

6.18. **Positive exponents.** For pot numbers where $n, r \geq 0$,

$$\overline{\infty} \times \overline{\infty} = (\overline{\infty})^2 \tag{33}$$

$$\overline{\infty} \times 1\overline{\infty} = \overline{\infty}^2 \tag{34}$$

$$1\overline{\infty} \times 1\overline{\infty} = \overline{\infty}^2 \tag{35}$$

$$x\overline{\infty} \times y\overline{\infty} = xy\overline{\infty}^2; \tag{36}$$

$$x\overline{\infty}^n \times y\overline{\infty}^r = xy(\overline{\infty})^{n+r} \tag{37}$$

For pan numbers where $n, r \geq 0$,

$$\infty \times \infty = \infty^2 \tag{38}$$

$$\infty \times 1\infty = \infty^2 \tag{39}$$

$$1\infty \times 1\infty = \infty^2 \tag{40}$$

$$x\infty \times y\infty = xy\infty^2 \tag{41}$$

$$x\infty^n \times y\infty^r = xy\infty^{n+r} \tag{42}$$

The next section will show there is no need to restrict pan numbers to positive exponents. The negative exponents are introduced separately to show the contrast with pot numbers.

6.19. **Negative exponents.** This section underscores the difference between inverses that are fractions and those that aren't. For pan numbers,

$$\overline{\infty} \neq \infty^{-1} \tag{43}$$

$$\overline{\infty}^{(-1)} = \frac{1}{\overline{\infty}} = \infty \tag{44}$$

$$\overline{\infty}^{(-n)} = \infty^n \tag{45}$$

$$\frac{1}{\infty} = \infty^{-1} \tag{46}$$

$$\frac{x}{\infty^n} = x\infty^{-n} \tag{47}$$

$$\infty^{-n} \times \infty^{-r} = \infty^{-(n+r)} \tag{48}$$

Equations 46 and 47 brings back questions regarding infinitesimals and the Archimedean axiom. Discussion of this topic and how to understand these numbers will be deferred to Section 8.2.

6.20. **Zero exponent.** As mentioned earlier, no change is proposed for the symbol"0" in its use as an exponent.

$$\overline{\infty}^0 = \frac{\overline{\infty}}{\overline{\infty}} = 1 \tag{49}$$

$$\infty^0 = \frac{\infty}{\infty} = 1 \tag{50}$$

6.21. **Mixing exponents.** Restatement of the basic multiplication relationship with implicit 1 made explicit.

$$\overline{\infty}^1 \times \infty^1 = 1 \tag{51}$$

The implication of the basic relationship is canceling. Exponents here indicate a regrouping; matching pots and pans in a one to one relationship insofar as is possible. Like musical chairs, whatever is still standing is the base.

$$\overline{\infty}^n \times \infty^n = 1 \text{ if } n \geq 0 \text{ and} \tag{52}$$

$$\overline{\infty}^n \times \infty^r = \overline{\infty}^{n-r} \text{ if } n > r \text{ and } r \geq 0, \text{ but} \tag{53}$$

$$\overline{\infty}^n \times \infty^r = \infty^{r-n} \text{ when } r > n \text{ and } n \geq 0 \tag{54}$$

In other words the exponent is $|n - r|$ and the base with the larger exponent becomes the new base (if $n \neq r$ and $n,\ r \geq 0$).

When $\overline{\infty}$ is in the dividend/numerator and ∞ is the divisor, then the quotient is $(\frac{1}{\infty})\overline{\infty}$. The Wallis number assumes the position of the Real part in the absence numbers. Given Equation 46 this may be written

$$\frac{\overline{\infty}}{\infty} = \infty^{-1} \times \overline{\infty} \tag{55}$$

In general, for all $n > 0$

$$\frac{\overline{\infty}^n}{\infty^r} = (\infty^{-r})\overline{\infty}^n \tag{56}$$

and is in final form.

As may be inferred from Equations 45 and 47, if $n < 0$, then Equation 56 resolves to pan numbers.

$$\frac{\overline{\infty}^n}{\infty^r} = \infty^{n-r} \tag{57}$$

6.22. **Roots and fractional exponents.** The usual rules for roots and fractional exponents apply.

$$\overline{\infty} = (\sqrt{\overline{\infty}})^2 \tag{58}$$

$$\infty = (\sqrt{\infty})^2 \tag{59}$$

$$\sqrt{\overline{\infty}^3} = \overline{\infty}^{\frac{3}{2}} \tag{60}$$

$$\sqrt[n]{\overline{\infty}^r} = \overline{\infty}^{\frac{r}{n}} \tag{61}$$

$$\sqrt[n]{\infty^r} = \infty^{\frac{r}{n}} \tag{62}$$

6.23. **Subtraction.** This section used to be much shorter. Originally, without thinking much about it, I just set $\overline{\infty}$ equal to $n-n$. Fortunately Terence Gaffney, a mathematician friend of mine, showed me the flaw in that. It leads to a logical contradiction at least as serious as the one with division by the other zero. So I fixed it. Then the gang at sci.math showed me the flaw in *that*. Here's my second fix based on their assistance.

6.23.1. *Subtracting Real numbers.* Anything minus itself equals the Wallis zero since it is unsigned. An example,

$$4 - 4 = \frac{1}{\infty}\overline{\infty}$$

In general,

$$x - x = \frac{1}{\infty}\overline{\infty} \tag{63}$$

This definition of zero replaces the historical, subtraction-based, definition of 0. The two, new definitions, both multiplicative and subtractive, indicate a team approach; a distribution of labor for the different aspects of zero. Utilizing the unsigned Wallis number and its zero turned out to be key in defining the subtraction of Real additive inverses and highlights the importance of an unsigned zero.

6.23.2. *Subtracting zero numbers.* Subtracting zero numbers leads to a squared zero.

$$x\overline{\infty} - x\overline{\infty} = (x - x)\overline{\infty} \tag{64}$$

$$= (\frac{1}{\infty}\overline{\infty})\overline{\infty} \tag{65}$$

$$= \frac{1}{\infty}\overline{\infty}^2 \tag{66}$$

6.23.3. *Subtracting panoply numbers.* Similar reasoning also applies to subtracting panoply numbers.

$$x\infty - x\infty = \frac{1}{\infty} \tag{67}$$

6.23.4. *Subtraction: quantity vs location.* A reminder that geometrical or locational arithmetic will be different. For example, $x - x = \frac{1}{\infty}$, the Wallis number instead of the Wallis zero. Again, there will be more on this in Sections 8.1 and 8.2.

6.24. **Addition rule of equality.** A question arises due to the definitions of the identity element of addition and the sum of additive inverses. With the current zero, the identity element, $x + 0 = x$, is easily shown equivalent to the sum of additive inverses by the additive rule of equality.

$$x - x + 0 = x - x \tag{68}$$

$$x - x + 0 = 0 \tag{69}$$

$$x - x = 0 \tag{70}$$

This is not the case with the new zeros.

$$x + n\overline{\infty} = x \tag{71}$$

$$x - x + n\overline{\infty} = x - x \tag{72}$$

and by Equation 63

$$x - x + n\overline{\infty} = \frac{1}{\infty}\overline{\infty} \tag{73}$$

$$x - x = \frac{1}{\infty}\overline{\infty} \tag{74}$$

Superficially, dropping the zeros looks the same in Equations 69 and 73. Each are just a simplification, a convention. But the latter is defined and leads to an apparent difficulty if followed through. To wit,

$$x - x + n\overline{\infty} - n\overline{\infty} = \frac{1}{\infty}\overline{\infty} - n\overline{\infty} \tag{75}$$

$$x - x + \frac{1}{\infty}\overline{\infty}^2 = \frac{1}{\infty}\overline{\infty} - n\overline{\infty} \tag{76}$$

by Equation 66. How is the "correct" result in the following equation

$$x - x = \frac{1}{\infty}\overline{\infty} \tag{77}$$

to be reached? At this time I must appeal to convention in the same way that it must be appealed to in Equations 69 and 70. This is not satisfying to me at present. More thought is needed. But note three things. One, this is a far less serious problem than the lack of definition of division by the other zero. But two, in Equation 76 notice the term $\frac{1}{\infty}\overline{\infty} - n\overline{\infty}$. Later in Section 8.2 we will see that it is desirable for the term $\frac{1}{\infty} - n$ to equal n. Which means that by Equation 29,

$$\frac{1}{\infty}\overline{\infty} - n\overline{\infty} = (\frac{1}{\infty} - n)\overline{\infty} \tag{78}$$

$$= n\overline{\infty}^2 \tag{79}$$

This result may be fine in and of itself, but not in relation to Equation 76. It is still better, I reiterate, than arithmetic without division by zero. And with the proper convention, it shouldn't be necessary to even go there. I hope. Oh, the third thing. All these proliferating zero numbers with exponents puts one's mind to Newton's Calculus. Instead of infinitesimals (or turtles), it's zeros all the way down.

6.24.1. *Reflexivity and identity.* Underlying the above difficulty is the oddity of different zeros sharing the same cardinality. Unlike fractions that share the same cardinality, the zeros cannot be reduced. At least not in the same sense. One thought I've had for dealing with this has to do with assigning somewhat different meanings to reflexivity and identity. For example zero numbers could be considered reflexive in that they share the same cardinality, but nonidentical when their Real parts are not equal.

6.25. **Order of operations.** PEMDAS, the order of operations, needs modifying to serve the needs of the new arithmetic. The new mnemonic peps things up. It works out to be PEPPMDAS. PP may indicate either pot/pan or pan/pot.

6.26. **Associate, commute, distribute.** Given the forgoing, the associative, commutative, and distributive properties hold no surprises. These properties continue within potential zeros and within panoply numbers. Furthermore, a field is formed by the new numbers together with the Reals. It may be termed a prairie to continue the sylvan theme of field and meadow, and to indicate the much larger scope of this field relative to them. This will be dealt with in a separate paper.

6.27. **Extended arithmetic: Reals, Pots, and Pans.** Adding and subtracting Real and totality number combinations is quite similar to adding and subtracting polynomials. Combine like terms and reduce. Multiply polynomial like expressions in the same manner as polynomials. The terms $\overline{\infty}$ and ∞ act much like the i in complex numbers or like variables in polynomials.

6.28. **Archimedean Axiom.** Some comment here to the effect that the Archimedean Axiom is not violated unlike the hyperreals?

7. The Nature and Meaning of Zero

Yet it is antecedently certain that continuities there must be; the new idea must be generated out of the old; it has its basis in them; and in the end its justification is found in the completion and organization which it contributes to them; – its removal of their surds and inconsistencies.

John Dewey

It is of the highest importance in the art of detection to be able to recognize, out of a number of facts, which are incidental and which are vital. Otherwise your energy and attention must be dissipated instead of being concentrated.

Sherlock Holmes

... axiom: Most improvements make matters worse.

George Will

This section is an exercise in retrograde analysis as is normal when introducing new axioms. Can a different notion of nothing, the placeholder hypothesis proposed in Section 3, be incorporated into the already very obvious basics of mathematical and even logical thought? If the potential zero substitutes well for the traditional, standard zero as I have attempted to show, then some foundational revision must lead to this number. To what depth need we go to change—if only a small tweak—the existing foundations of the zero edifice? And the change should be relatively innocuous lest cracks appear in the foundations of other parts of mathematics and violate the Principle of Permanence. After all, we want a different nothing that enriches rather than harms mathematics.

My analysis covers two areas. One is the axioms of Richard Dedekind familiar to many as the Peano Postulates without the zero postulate. The other area is the axioms of Ernst Zermelo, Thoralf Skolem, and Abraham Fraenkel, the basis for Zermelo–Fraenkel set theory. In both cases I follow precedents given by Giuseppe Peano and Gottlob Frege in the foundations for the standard zero as much as possible. Like Peano, I will add a zero axiom to the Dedekind axioms. In Frege's case, I reject the empty set in favor of another set. The continuity with his work lies in using the reflexive relationship as part of the basis for a set of nothing. Both the empty set and my alternative, *the absent set*, can be derived using the reflexive property. Once the absent set is established, its substitution in the Zermelo–Fraenkel axioms for the empty set is addressed.

Adding a zero axiom to the Dedekind axioms is pretty straightforward. There are a few wrinkles, but nothing major. However, my alternative to the empty set is not so straightforward. Frege's basis for the empty set is in terms of first order logic together with the reflexive property of equality. The modification I introduce (which I would like to think is in the way of a clarification) involves a subtle point concerning the existential quantifier. The point turns out to depend on what the definition of "is" is. Once understood, my amendment to his quantifiers leads directly to the absent set.

As a general comment, this section will be limited primarily to issues pertaining to the mathematics of nothing; its logic and the symbols used. However, contrasts between Frege's work and mine call into question some fundamental views of reality that are untestable in principle. Reference to such fundamentally different views and their consequences has already been made toward the end of Section 4 in terms of metaphysics in Popper's sense and will be addressed in Section 10. The contrasts also pertain to the issues regarding empiricism mentioned at the end of Section 2. These issues are addressed explicitly here in Section 7.3.1.

7.1. **An alternative standard arithmetic.** Peano and I both share the view that nothingness is a basic and obvious mathematical concept deserving of its status as an axiom. We differ as to the symbol and the concept of nothing it represents.

Axiom. $\overline{\infty}$ *is a number.*

The difference arises from the different primitive terms derived as they are from different parts of arithmetic, "—" from division and "0" from subtraction. Another obvious difference will appear below when the axiom is placed in a different order than Peano's.

For those who may argue that the new zero is sufficiently different and thus improperly classed as a number, please feel free to substitute "array" or something to the effect of "manipulable mathematical object or term" for "number" in the axiom.

7.1.1. *Primitive terms.* Two primitive terms, "—" and "∞," "replace the primitive term "0." The line is called the absence bar. Its antecedent is the bar used in fractions. It indicates the absence of whatever is beneath the bar. This component is crucial for the purposes of arithmetic with the new zero and should not lightly be changed. And it is crucial for the potential zero interacting with other numbers in ways the traditional zero cannot.

The infinity symbol, on the other hand, is less crucial. I do find this choice desirable for a number of reasons, but other symbols may be used and in some particularly complex instances even be decidedly preferable. Whichever symbol is used, it should indicate the universe of discourse in use (or possibly some subset). In the case of standard arithmetic the specified numbers are the Natural numbers. How that specification is introduced axiomatically will be considered next.

7.1.2. *Placing the zero axiom.* Unlike with the zero axiom in standard arithmetic, the alternative zero axiom does not precede the others. Since it has to do with the absence of the Natural numbers, it must needs be placed after the axioms establishing those numbers. It is also necessary to define ∞ as the Natural numbers. So a statement such as $\mathbf{N} \equiv \infty$ should be placed between the successor axioms and the zero axiom. Here **N** should not be considered to possess cardinality or any other attribute except as given in the successor axioms. **N** is a simple series as stated in Section 5.4.

7.1.3. *Dedekind/Alternative Axioms.* As mentioned earlier, I will be following the same procedure as Peano; using Dedekind's axioms and adding axioms for zero. This is an alternative to Standard Arithmetic.

The following is based on Eric Weisstein's entry for Peano's Axioms in Wolfram Research's Mathworld™.[59]

Peano's Axioms, revised version:

(1) One is a number.
(2) If a is a number, the successor of a is a number.
(3) One is not the successor of a number.
(4) Two numbers of which the successors are equal are themselves equal.
(5) (induction axiom.) If a set S of numbers contains one and also the successor of every number in S, then one and every successive number is in S.
(6) S is a number and is represented by ∞.
(7) One $\overline{\infty}$ (or $1\overline{\infty}$) is a number.
(8) If $a\overline{\infty}$ is a number, the successor of $a\overline{\infty}$ is a number.
(9) One $\overline{\infty}$ is not the successor of a number.

[59]Weisstein, Eric W. "Peano's Axioms." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/PeanosAxioms.html`

Instead of five axioms there are now nine. Axiom 4 applies to the zero numbers and does not need repeating. Induction applies as well. The additional axioms are justified by a gain in explanatory power.

7.2. **Replacing the empty set.** The empty set provides a basis for the traditional number zero from the standpoint of set theory. It needs replacing in set theoretic axioms in order to produce an emptiness consistent with the potential zero. A represention for this notion called the *absent set* is introduced below. The logical basis for this represention has some commonalities with Frege's basis for the empty set. We'll turn to his work as a starting point.

7.2.1. *Frege's logic.* Here's an example of the ingenious way Gottlob Frege arrived at the empty set. He makes use of reflexivity or self-identity.

> We choose a set M, and let $\mathcal{A}(x)$ be the condition (statement) $x \neq x$. Then there is a set $\mathcal{A}$ which consists of those elements $x \in M$ for which $x \neq x$. This set is denoted by $\emptyset$ and is called the *empty set.*[60]

Let's also review its counterpart, the equality $x = x$. In the same context, an element of M is observed in the expression $x = x$. In this case, each element satisfies the equality condition. The element is then placed in set $\mathcal{A}$.[61]

Alfred Tarski(using different terminology) comments that of the sets formed from $x = x$ and $x \neq x$, the "first ... is satisfied by every individual and the second by none."[62] I have no argument with any of this if, that is, one wants only the standard zero. And so it is amidst this material that I propose to create an absent set.

7.3. **The absent set.** Of crucial importance for establishing the absent set is the meaning assigned to the term "is" in the phrase "there is." The possible definitions are in the dictionary under the word "be." There are several. Frege's choice is made clear by his use of "there exists" interchangeably with "there is." This is, of course, the existential

[60] *Oxford Users' Guide to Mathematics*, Zeidler, E., ed., Hunt, B., trans. Oxford University Press, 2004, p. 901. Title of the original: *Teubner-Taschenbuch der Mathematik, Vol 1*, 1996.

[61] The disposition of M is not normally discussed as far as I know and it is not relevant for my purposes. But I wonder, does it empty out? Or is it still full of all its elements? Maybe perfect copies of all the elements end up in $\mathcal{A}$.

[62] Tarski, Alfred, and Jan Tarski, *Introduction to Logic and to the Methodology of the Deductive Sciences*, 4th ed., Oxford University Press US, 1994, p. 67.

quantifier the symbol for which is "$\exists$."[63] What if a different meaning for "is" is used?

Another common definition of "is" is: "To have, maintain, or occupy a place, situation, or position<the book *is* on the table>."[64] Based on this definition, I would like to introduce the phrase "here is" into first order logic. Let's start to revise Frege by substituting "here is" in the example above.

> We choose a set M, and let $\mathcal{A}(x)$ be the condition (statement) $x = x$. Then *here* is a set $\mathcal{A}$ which consists of those elements $x \in M$ for which $x = x$.

Saying "here is a set 'satisfied by every individual' " makes it possible to refer to a set that isn't here, but could be. By agreeing that the negation of "here," has the sense of "being absent" or "is absent," first order logic will have two additional choices: presence and absence. To put it another way, simple existence would be defined in terms of *here* and *not here* to go along with lack of existence.

For a more formal symbol, I propose the letter "H" to symbolize the quantifier "here is" in the same sans serif font as $\exists$. The hereness or *presential* quantifier, H, so as to be in keeping with the tradition of the other quantifiers, should be considered both reversed and inverted.

The absent set follows from the negation of this quantifier.

> We choose a set M, and let $\mathcal{A}(x)$ be the condition (statement) $x = x$. Then a present set $\mathsf{H}\mathcal{A}$ is *not* here which consists of those elements $x \in M$ for which $x \neq x$. This set is denoted by $\not\mathsf{H}\mathcal{A}$ and is called the *absent set*.

Putting $\not\mathsf{H}$ in front of any set designates that set as absent. Using "$\not\mathsf{H}$" or "$\neg\mathsf{H}$" indicates the *absential* quantifier. It seems rather messy and I welcome someone devising a symbol as elegant and evocative as "$\emptyset$" is for the empty set. Or perhaps it won't seem so messy after a while.

The empty, or null set, instead of being derived as in $\exists x \neg\exists y (y \in x)$,[65] would be written $\mathsf{H}x \neg\exists y (y \in x)$ where exists is a shorthand for present and absent, $(\mathsf{H} \vee \neg\mathsf{H}) \Leftrightarrow \exists$.

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[63] Sometimes "for some" is used instead to express $\exists$. See *The Princeton Companion to Mathematics*, Timothy Gower, ed., p. 14. This makes no difference to the subject at hand.

[64] Merriam-Webster, Inc, *Merriam-Webster's Collegiate Dictionary*, 11th ed., Merriam-Webster, 2003, p. 105.

[65] Jech, Thomas, "Set Theory", *The Stanford Encyclopedia of Philosophy* (Spring 2009 Edition), Edward N. Zalta (ed.), `http://plato.stanford.edu/archives/spr2009/entries/set-theory/ZF.html`

7.3.1. *Empirically friendly sets.* Absent and present sets are empirically friendly because presence and absence are matters of observation. "Something exists" is simply an assertion which is of itself independent of sensory data. It lacks cognizable content. Existence should be determined by observation. If we understand "exists" as meaning something either determined to be present(sensed) or absent(but capable of being sensed), then the empty set would consist of those members that don't exist because they are neither present nor capable of being present. This actually describes the empty set as defined by Frege pretty well. An element x defined by $x \neq x$ can neither be present nor capable of being present. If only he'd reconsidered $\exists x$ from this standpoint. . ., but he didn't.

The argument I am making about empirically friendly sets applies to objects available to the senses. What about mental objects like the empty set? I think it is not too much of a stretch to make the general thrust of the argument apply to them as well. Simply take into account the mentally cognizable along with the sensibly cognizable. I refer again to the quote by Peirce at the end of Section 4, especially the part where he says "we cannot conceive the *real* to be anything other than that which is *cognizable*."[66] And would say that this applies to the *existent* just as much as to the *real*. In other words, whatever members may be contained in the domain of discourse, those members' existence is equivalent to their being cognizable sensibly or mentally.

As to how the domain of discourse may be enlarged or modified, I tend to the thought of John Dewey.

> And so it is with mathematical knowledge, or with knowledge of politics or art. Their objects are not known till they are made in course of the process of experimental thinking. Their usefulness when made is whatever, from infinity to zero, experience may subsequently determine it to be.[67]

Dewey is at pains to say that he means experience in a very unrestricted, encompassing way. But this takes us rather far afield.

The preceding lays out a basis for an alternative to the empty set in terms of logic. I now turn to showing how this alternative might apply specifically to mathematics by describing how some of the Zermelo–Fraenkel axioms can be modified. The basic approach should provide a good indication as to how axioms in other systems can be modified.

[66]Apel, p. 11.

[67]Dewey, John, *Essays in Experimental Logic*, Dover Edition, 2004, p. 213.

7.4. **Modifying Zermelo–Fraenkel set theory.** Various versions of the Zermelo–Fraenkel axioms exist. A helpful overview by Eric Weisstein is at Wolfram Research's Mathworld$^{\mathrm{TM}}$. The following is in reference to his entry.[68]

The presential quantifier introduced in Section 7.3 and the existential quantifier are interchangeable in Zermelo–Fraenkel set theory. Practically speaking, if something exists, then it's here, or, perhaps, vice versa. Both denote "for some" so nothing more is needed. Updating ZF by substituting one for the other is a simple matter.

The other step in modifying the axioms of Zermelo and Fraenkel pertains to the empty set. It occurs in two places: the axioms of Infinity and Foundation. In the case of the Axiom of Foundation (also called Axiom of Regularity) there is no change. The empty set stays. It refers to something neither present nor absent in this axiom as can easily be seen.

Axiom of Foundation. $\forall S[S \neq \emptyset \Rightarrow (\mathsf{H}x \in S) S \cap x = \emptyset]$

For the Axiom of Infinity, a return to the tradition of Dedekind's axioms is recommended. The empty set should be replaced with a first element such as used in counting. As Dedekind says "We call this element, which we shall denote in what follows by the symbol 1, the *base-element* of N..."[69] So the Axiom of Infinity could be written

Axiom of Infinity. $\mathsf{H}S[1 \in S \wedge (\forall x \in S)[x \cup \{x\} \in S]]$

As has become common, the empty set is folded in to the Axiom of Infinity instead of being established by its own axiom. For clarity, the axiom for the absent set will be separate. The separateness should help comparison of the differing axiom systems. Besides that, it just doesn't seem to belong there.

Axiom of the Absent Set. $\neg\mathsf{H}S[(\forall x \in S)[\neg\mathsf{H}x]]$

It may also be written

Axiom of the Absent Set. $\not\mathsf{H}S[(\forall x \in S)[\not\mathsf{H}x]]$

I have felt rather out of my depth in this section, but hope some effort made here will be helpful in choosing an axiom or axioms for the potential zero.

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[68]Weisstein, Eric W. "Zermelo-Fraenkel Axioms."From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/Zermelo-FraenkelAxioms.html`

[69]Dedekind, p. 33.

8. Some Alternative Mathematics

> *The mathematician is embarked on an adventure which he can only stop in an arbitrary way and which at every instant brings a radical novelty.*
>
> *Jean Cavaillès*

> *In just such investigations one needs to exercise the greatest care so that even with the best intention to be honest he shall not, through a hasty choice of expressions borrowed from other notions already developed, allow himself to be led into the use of inadmissible transfers from one domain to the other.*
>
> *Richard Dedekind*

[Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

8.1. **The Real Number Line.**

> *Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone.*
>
> *Richard Dedekind*

[Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

`This generalization (from Reimann sphere`
`to plane/number line since substitute Wallis number)`
`will be important in Section 8.1`
`for straightening out issues relating to the origin`
`of the Real Number Line and in Section 8.4`
`for like issues with the origin`
`of the coordinate plane and`
`will be dealt with there.` Zero not on number line. Potential zero is absence of point sets so is not a point set itself. Origin a different number(see next section). Riemann magnitude–Dedekind divorced number definitions from magnitude; zero as magnitude –zero as number

Differences in arithmetic relate to difference between quantity and magnitude.

Div by zero orthogonal rotation = line set topology

8.2. **The Wallis Number.** [Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

∞^{-1} and $\frac{1}{\infty}$ New origin number—substitute for zero on number line.

8.3. **Trigonometry.** [Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

Tangent = ∞ — points at orthogonal rotation for div by zero.

zero degrees = $\overline{\infty}$ degrees

8.4. **Geometry.** [Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

origin of coordinate plane = ∞^{-1} and $\frac{1}{\infty}$ – the Wallis number again.

Slope of vertical line equals ∞

Div by zero orthogonal rotation = helping to create plane

Plane created/*constructed* by $\infty \times \infty = \infty^2$

8.5. **Exponents as Dimension.** [Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

`separate subsection? -- zero vs zeroth dimensions, zero D as identity dimen. Remember Wheeler ref in Road to Reality` Division by zero is a change of dimension or an orthogonal rotation of the totality ∞. Section 6.17 generalizes this for repeated divisions by zero. Although not apparent here, I should also note some similarity to a notation used by John Archibald Wheeler when exponents are used with the infinity symbol. I came across a brief reference recently[70], but have not yet been able to gain access the original and investigate just how similar the notations are.

Perhaps generalize to nonorthogonal rotations. Vectorlike.

`comment: remember to discuss identity one! Give it a name?`

[70]In Roger Penrose's Road to Reality, p. 380

8.6. **The Calculus.** [Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

Lots less discontinuities. Much of calculus is now arithmetic.

Refer to exception to order of operations in calculus.

Berkeley answered. Similarity to Newton's original calculus.

`Archimedian axiom; Dedekind postulate re sections per Cajori on p. 35`

8.7. **Complex Numbers.** [Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

Seems to work for complex numbers, too. Absence of complex numbers made present through division by zero. Is there an imaginary zero; $\overline{\infty}\, i$? If on plane –yes, if not no? Is $\frac{1}{\infty}$ imaginary, but not $\overline{\infty}$?

This notation, 4∞, is very similar to one for a directed infinity.[71] It is used for results from computations on the Complex numbers and the Complex plane in the software package Mathematica.

There is a traditional symbol for complex infinity[72] (Riemann's point at infinity) that looks similar to the potential zero.

Dividing $1/0 = \infty$ in Complex numbers is defined.[73] BUT $1 \neq 0 \times \infty$.

8.8. **Matrices.** [Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

The multiplicative inverse of a nonsingular matrix is its matrix inverse.[74]

`An array is a one line matrix?`

Linear algebra the following is a quote from somewhere *In matrix algebra (or linear algebra in general), one can define a pseudo-division, by setting* $a/b = ab^+$, *in which* b^+ *represents the pseudoinverse of* b. *It can be proven that if* b^{-1} *exists, then* $b^+ = b^{-1}$. *If* b *equals* 0, *then* $0^+ = 0$*; see Generalized inverse.*

[71] From MathWorld–A Wolfram Web Resource. `http://functions.wolfram.com/Constants/DirectedInfinity/introductions/Symbols/02/`

[72] Weisstein, Eric W. "Complex Infinity." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/ComplexInfinity.html`

[73] Weisstein, Eric W. "Division by Zero." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/DivisionbyZero.html`

[74] Barile, Margherita. "Multiplicative Inverse" From MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. `http://mathworld.wolfram.com/MultiplicativeInverse.html`

9. Some Applications to Physics

The answer to these questions [about the reality which underlies space] can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices. This leads us into the domain of another science, of physic, into which the object of this work does not allow us to go to-day.

Bernhard Riemann

I would remark in passing that a recognition that a thing may be continuous in one respect and discrete in another would obviate a great many difficulties.

John Dewey

All things prevail for those for whom emptiness prevails; Nothing whatever prevails for those for whom emptiness prevails.

Nāgārjuna

[Under Construction] [Remaining sections consist of rough notes except for Section 9.1 on the use of the Lorentz term in Einstein's Special Theory of Relativity.]

The purpose of this section is to look at some examples where zero is important in physics. Differences in mathematics with the potential zero are presented in order to call attention to cases where it may be possible to see if the new "number-domain created in our mind" helps to more "accurately investigate our notions of space and time."

9.1. **The Lorentz term and the potential zero.** The Lorentz factor or term is a basic building block of Einstein's Special Theory of Relativity. It appears in equations such as those for length contraction, time dilation, and relativistic mass. Equation 80 is a version of the Lorentz term.

$$\gamma = \frac{c}{\sqrt{c^2 - u^2}} \text{ where u=velocity} \tag{80}$$

The result when the velocity u equals light speed, c, is the math underlying the well known idea that the speed of light is a speed limit. Although undefined mathematically, division by 0 is interpreted here as infinity in the sense of increasing without bound. What happens when a new zero is used? Here the simplest case of the potential zero is used.

$$\gamma = \frac{c}{\sqrt{\overline{\infty}}} \tag{81}$$

$$\gamma = \frac{c}{\overline{\infty}^{1/2}} \tag{82}$$

$$\gamma = c \cdot \infty^{1/2} \tag{83}$$

$$\gamma = c\,\infty^{1/2} \tag{84}$$

Division with this new zero is defined as has been shown earlier. Is its definition in accord with the existing experimental evidence of physics? Or might it point toward a new avenue of inquiry?

Division with the new zero indicates that γ is at an inflection point. One way to read this is that $c\,\infty^{1/2}$ indicates some sort of rotation. The math may indicate a transformation of the energy of the velocity into something else. Perhaps it can be used to describe a physical process like complex numbers describe the transformation of electricity and magnetism from one to the other. In any event the math is quite different and may warrant a fresh look at the phenomena.

Questions arise as to whether a zero of dimension one is the appropriate one to use here. Determining what is absent is necessary for deciding the dimensionality of the difference. Might it be more appropriate to say that the absent is of the four dimensions of spacetime? If so, then the γ would be equal to an expression raised to the 2nd power.

$$\gamma = c\,\infty^{2} \tag{85}$$

Some thought seems to be needed here.

Along with the dimensionality (correct exponent) of the difference, it may be necessary to decide the type of difference as well. Is a zero or a Wallis number more appropriate? However, the results in Equations 84 and 85 would stay the same regardless of the choice made.

9.2. **Singularities Inflected.** [Under Construction]
[Remaining sections consist of rough notes]

Current view - physics disappears.

`Dr. Wheeler at first resisted this conclusion, leading to a confrontation with Dr. Oppenheimer` at a conference in Belgium in 1958, in which Dr. Wheeler said that the collapse theory does not give an acceptable answer to the fate of matter in such a star. He was trying to fight against the idea that the laws of physics could lead to a singularity, Dr. Charles Misner, a professor at the University of Maryland and a former student, said. In short, how could physics lead to a violation itself to no physics?

Dr. Wheeler and others were finally brought around when David Finkelstein, now an emeritus professor at Georgia Tech, developed mathematical techniques that could treat both the inside and the outside of the collapsing star.

quoted from NYT Obituary

Comment: previous section leads into inflection points in GR, i.e. Big Bang, and in general.

9.3. **The Dirac delta function.** [Under Construction]
[Remaining sections consist of rough notes]

Paul Dirac needed a function that does not exist in the mathematics he knew. He needed an integral, or area, of an interval when the interval becomes zero. The need addressed by the Delta Function is built into the new arithmetic. When an interval of an integral is equal to zero essentially the need is to find the area of a line (or ray) with width zero. As discussed in Sections 8.1 and 8.2, the width of a line is the Wallis number, $\frac{1}{\infty}$. The Wallis number applies because the line is centered on a point. The length times the width is $\frac{1}{\infty} \times \infty$ and equals one.

> You can visualize the delta function as a function that vanishes outside a narrow interval on the x-axis, and whose integral over any region that includes that interval is 1. But the delta function cannot be defined as the limit of such a function – call it Dn – as the narrow interval approaches 0, because the limit does not exist. What does exist, though, is the limit as n approaches infinity of the integral of the product of Dn and any ordinary function f(x). If Dn is centered on the point a, the limit of the integral of this product approaches f(a). This is the context in which the delta "function" appears in all applications.[75]

[75]Layzer, David, personal communication, paraphrasation of passage from Dirac, Paul, "The Principles of Quantum Mechanic" 4th ed. A clear account of the delta function begins on p. 58.

9.4. **Space: grainy or smooth?** [Under Construction]
[Remaining sections consist of rough notes]

> *If any one should say that we cannot conceive of space as anything else than continuous, I should venture to doubt it and to call attention to the fact that a far advanced, refined scientific training is demanded in order to perceive clearly the essence of continuity and to comprehend that besides rational quantitative relations, also irrational, and besides algebraic, also transcendental quantitative relations are conceivable.*
>
> *Richard Dedekind*

If we suppose that bodies exist independently of position, the curvature is everywhere constant, and it then results from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes may be neglected. But if this independence of bodies from position does not exist, we cannot draw conclusions from metric relations of the great, to those of the infinitely small; in that case the curvature at each point may have an arbitrary value in three directions, provided that the total curvature of every measurable portion of space does not differ sensibly from zero. Still more complicated relations may exist if we no longer suppose the linear element expressible as the square root of a quadric differential. Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.

The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must

> come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the gound of its metric relations outside it, in binding forces which act upon it.
>
> The answer to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices.
>
> This leads us into the domain of another science, of physic, into which the object of this work does not allow us to go to-day.[76]
>
> *Riemann*

`Comment: The zero's universe, whether continuous or only everywhere dense, may hold some relevance to this question.`

`dialetheism`

Absence may provide continuity to (discontinuous) presence.

9.4.1. *Virtual Particles: Out of the Zeroth Dimension?* [Under Construction]

[Remaining sections consist of rough notes]

> *And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desired, from filling up its gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle.*
>
> *Richard Dedekind*

`do they come from zeroth dimension? Sure, why not?`

10. A philosophy of no thing

> *There is no such thing as nothingness, and zero does not exist. Everything is something. Nothing is nothing. Man lives more by affirmation than by bread.*

[76]Riemann, pp. 7-8.

Victor Hugo

Set theory is exact theology.

Rudy Rucker

[Under Construction] [Remaining sections consist of rough notes]

Remember reference to Popper's metaphysical research programme earlier and that this is support for placeholder hypothesis as mentioned end of section 2.

The criteria of contingency and relationship matches the view of nothings found in a particular Indian philosophy. This philosophy was put forth in the 2nd and 3rd Centuries C. E. by the influential Buddhist reformer Nāgārjuna. Note that a wide range of interpretation and understanding of his work has developed over the centuries. Nāgārjuna's thought has been likened to that of such disparate figures as Kant, Hume, Nietzsche, and Wittgenstein. To illustrate the variety further, here's a look at some of the translations of a key term, sunyata. Examples include emptiness, relativity, fullness, and void. Naturally, the interpretations presented here support the ideas related to the new zero. Other interpretations may not be so supportive. Also note that the terms nothingness and emptiness are used interchangeably for the most part. Context makes clear the couple of places where they differ.

Nāgārjuna states the following[77] about emptiness

> All this is empty

The key word here is this. Professor David Kalupahana comments on the significance of Nāgārjuna's statement.

> Thus, for Nāgārjuna, emptiness(sunyata) was *no more* than what is implied in the statement "All this is empty." The statement "All this is empty," is not identical with the statement "All is empty." In fact, as pointed out in the annotation, nowhere in the Karika[78] can one come across an absolute statement such as "All is empty." It is indeed significant that even when making a universalized statement, Nāgārjuna retains the demonstrative "this" in order to eliminate the absolutist sting.[79]
> (italics in original)

As Kalupahana makes clear, Nāgārjuna's hypothesis concerning emptiness(sunyata) is always and only made up of particular instances of "the

[77]Quotation from Mulamadhyamakakarika of Nāgārjuna - The Philosophy of the Middle Way, Kalupahana, David, p. ??.

[78]Karika is a common abbreviation for Nāgārjuna's treatise.

[79]Kalupahana, p. 85.

empty(sunya)." In other words, this emptiness, or nothingness, is always contingent and dependent upon specific "nothings" such as those given in the preceding two sections

This notion of nothingness dependent upon specific nothings is in contrast to the notion underlying the number zero presently found in mathematics. That notion is absence only and is not dependent on anything – certainly not on the counting numbers. A symbol well suited for holding a place has simply been plopped down amongst the other numbers. It works well for addition and subtraction; not so well for multiplication and division.

If we accept that the current number zero is not dependent, at least one more possibility arises. Can the current zero itself be regarded as a specific instance of the empty? After all, the word zero does derive from sunya, the Sanskrit word for empty used by Nāgārjuna. The key question for Nāgārjuna would be "Is it evident?" Does it have a referent in an empirical sense? Does it actually refer to an "absence only" that is unaffected by things? Nāgārjuna might reason like this: If absence only exists then it either has mathematical bounds or it does not. If it has bounds, it is dependent upon bounds, and thus is not only absence. If it does not have bounds it would be the only mathematical object. However, other mathematical objects exist, therefore zero, in the sense of absence only as presently hypothesized, is not evident and therefore not a specific instance of the empty.

My reasoning a la Nāgārjuna is supported, I believe, by Kalupahana's analysis of the importance of evidence, identification, and empiricism in Nāgārjuna's thought. A brief example should suffice.

> An absolutistic view of emptiness would certainly contradict [Nāgārjuna's] empirical method that calls for identification as a test of truth or reality. "Non-substantiality" or "emptiness," taken in themselves, would be as abstract and unidentifiable as a substance.[80]... "emptiness," distinguished from "the empty"... would be as unidentifiable and therefore nonsensical as any other metaphysical conception that Nāgārjuna was endeavoring to refute.[81] (quotation marks in original)

I am reading the "absolutistic view of emptiness" referred to above as emptiness that exists without reference to anything else. This seems to me to describe the current zero. It is an " 'emptiness,' distinguished

[80]The substance Nāgārjuna refutes refers to any metaphysical or non-empirical substance.

[81]Kalupahana, p. 85.

from 'the empty.' " In other words the primitive term zero as presently hypothesized can be considered as absence in and of itself. It is not in reference to some collection of specific absences of numbers considered as a totality. To be sure, the *number* zero interacts with other numbers through some of the rules of arithmetic, but this does not affect the assumptions regarding absence found in the primitive term as instituted by the independent zero axiom.

Finally, I wish to bring out one more point from Nāgārjuna's thought of relevance to the new zero. That is a sense in which potential is related to nothingness. To do so we turn from the close scholarly analysis of Professor Kalupahana to a philosopher, Daisaku Ikeda, writing about Nāgārjuna's emptiness from a completely different Buddhist tradition.

> Nothing can be born out of mere [nihilistic] nothingness. But from the "emptiness" of the Middle Doctrine, which is a kind of infinite potentiality, anything and everything may be born or produced, depending upon what causes happen to affect it. Various objects and phenomena appear to the ordinary beholder to be arising out of nothing. But what precedes them is not in fact [nihilistic] nothingness but the state of *ku* [Jpn.] or potentiality that Nāgārjuna has been describing.[82]

This passage shows, yet again, the deep contrast between the competing versions of nothing. This concludes my discussion of the various ideas, notions, and concepts underlying the potential zero. An example of a number zero based on them is introduced in the next section.

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[82]*Buddhism, the First Millenium*, Ikeda, Daisaku, trans., Watson, Burton, 1977, p. 141.