2.6.5 *What the infalling observer feels*

Seen from a local frame fixed with respect to the Schwarzschild coordinates a radially infalling particle having $E = 1$ falls with speed $v_{\text{loc}} = (2m/r)^{1/2}$ (section 2.30, or Eq. (2.33)). If we work from the frame of the freely falling particle, the speed of the coordinate lattice in this frame is also $v_{\text{loc}} = (2m/r)^{1/2}$. Let τ be the proper time of the particle. The acceleration of the stationary frame is $dv_{\text{loc}}/d\tau = (dv_{\text{loc}}/dr)(dr/d\tau) = (dv_{\text{loc}}/dr)v_{\text{loc}} = m/r^2$.

As in Newtonian physics the dependence of the acceleration on radial distance means that an extended body will feel a tidal force. We want to calculate this tidal force from the point of view of the infalling observer (who will after all be subject to the force). We follow the derivation in Taylor and Wheeler (2000). Any two observers agree on their relative acceleration, so the relative acceleration of the head and the feet of a radially aligned infalling observer is

$$dg = \frac{2m}{r^3}dr.$$

To get the tidal force we need the proper distance corresponding to a coordinate distance dr (because our height that we measure in metres is our proper height in free fall, not, in principle, a coordinate displacement in Schwarzschild coordinates, although the two will turn out to be numerically the same).

In a stationary frame the proper distance corresponding to a displacement dr is $dr\,(1 - 2m/r)^{-1/2}$. For the freely falling observer moving with speed $v_{\text{loc}} = (2m/r)^{1/2}$, this length is contracted by a factor

$$1/\gamma = \left(1 - v_{\text{loc}}^2\right)^{1/2} = (1 - 2m/r)^{1/2}.$$

So the proper length in the freely falling frame is

$$\gamma^{-1}dr\,(1 - 2m/r)^{-1/2} = dr.$$

The tidal acceleration on a body of extension Δr is therefore approximately $(2m/r^3)\Delta r$. At some point this will begin to cause pain for the infalling observer. We want to calculate how long this pain must be endured before the singularity at $r = 0$ is encountered.

Suppose that our criterion for the onset of pain is $\Delta g = g_{\text{E}}$, the gravity at the Earth's surface. This will occur at a radius $r_p = (2m\Delta r/g_{\text{E}})^{1/3}$. The infall time to $r = 0$ is obtained by integrating (2.30):

$$\tau = \int_{r_p}^{0} \left(\frac{r}{2m}\right)^{1/2} dr = \frac{2r_p^{3/2}}{3(2m)^{1/2}} = \frac{2}{3}\left(\frac{\Delta r}{g_{\text{E}}}\right)^{1/2}.$$

Note that this is independent of the black hole mass.