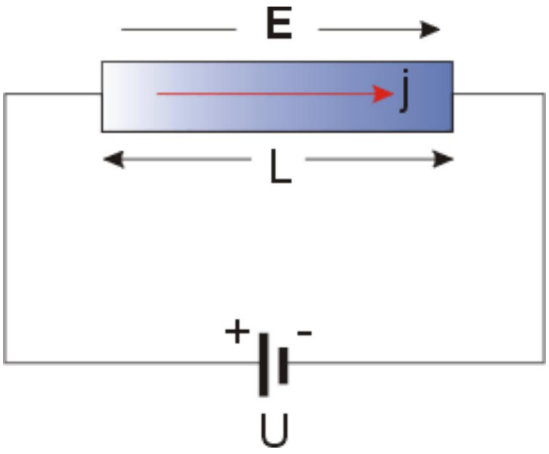
CONDUCTION

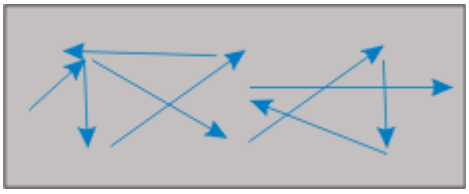
Original source: http://www.physik.uni-stuttgart.de/ExPhys/2.Phys.Inst./official/m.mehring/CondMat/conduction.htm

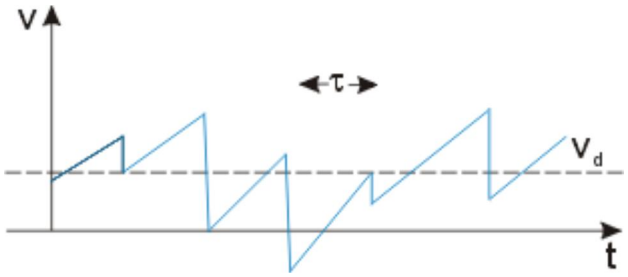


Electrical conduction of metals is a well known phenomenon and is attributed to the rather free conduction electrons. It is measured as sketched in the figure. The current density j is observed to be proportional to the applied electric field and follows Ohm's law $\mathbf{j} = \sigma \cdot \mathbf{E}$ where the prefactor is the specific conductivity. Since the electric field and the current density are vectors we have expressed Ohm's law here in bold face. The conductivity can in general be expressed as second rank a tensor (3x3 matrix). Here we restrict the discussion to isotropic, i.e. scalar conductivity. The specific resistivity $\rho = 1/\sigma$ is the inverse of the conductivity. Both parameters will be used in the following. Drude (at about 1900) already realized that the phenomenological description of conductivity can be formulated quite generally (electron-, ion-, heat- etc. conductivity). Although the phenomenological description is incorrect for conduction electrons, it can serve as an introduction to the more general treatment. The assumption is that the electrons move freely in the solid like in an ideal gas. The force applied to the electron by the electric field leads to an acceleration according to

$$\mathbf{F} = -e\mathbf{E} = m\frac{d\mathbf{v}}{dt}$$
$$\Rightarrow d\mathbf{v} = -e\mathbf{E}dt$$

This would lead, however, to an infinite velocity. The further assumption therefore is that the electrons bump into obstacles (like defects or phonons) once in a while which limits their free flight as shown in the next figures.





This establishes an average or *drift velocity* v_d. This drift velocity is related to the average scattering time as becomes evident from the following relations.

$$\frac{dv}{dt} = -\frac{e}{m}E - \frac{1}{\tau}v \Rightarrow \text{ equilibrium with } \frac{dv}{dt} = 0 \Rightarrow v_d = -\frac{e\tau}{m}\cdot E$$

This relation holds in general for classical particles with charge q and mass m which allows us to express the drift velocity and the corresponding mobility as

$$v_d = \mu E = \frac{q\tau}{m}\cdot E \quad \text{with mobility} \quad \mu = \frac{q\tau}{m} \quad \text{in units of} \quad \left[\frac{m^2}{Vs}\right].$$

Ohm's law can now be reformulated in terms of these phenomenological parameters as derived by Drude

$$j = n \cdot q v_d \Rightarrow j = n \cdot q \cdot \mu \cdot E = n \frac{q^2 \tau}{m} \cdot E = \sigma E$$

$$\Rightarrow \sigma = n \cdot q \cdot \mu = n \frac{q^2 \tau}{m} \xrightarrow{q=e} n \frac{e^2 \tau}{m}$$

This allows to summarize the specific conductivity and resistivity in the Drude form as

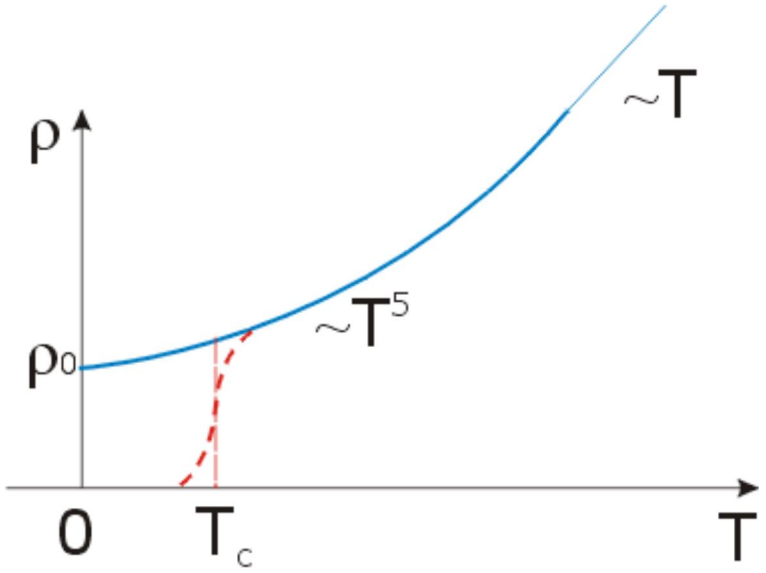
$$\sigma = n \frac{e^2 \tau}{m} \Rightarrow \rho = \frac{m}{n e^2} \cdot \frac{1}{\tau}$$

With typical parameters for the electronic charge and mass and a number density $n=N/V$ like one per atom the typical metallic conductivites of 10^7 $Ohm^{-1}m^{-1}$ are obtained for a scattering time of 10^{-13}s. The Drude model does not only describe the order of magnitude of the conductivity successfully it also allows to explain the temperature dependence. This is seen from the following expressions.

With $\frac{1}{\tau_{phonon}} = \begin{cases} \sim T^5 & T < T_D \\ \sim T & T \geq T_D \end{cases}$ the total scattering rate

$$\Rightarrow \frac{1}{\tau} = \frac{1}{\tau_{defect}} + \frac{1}{\tau_{phonon}} \Rightarrow \rho = \frac{m}{n e^2} \left(\frac{1}{\tau_{defect}} + \frac{1}{\tau_{phonon}} \right)$$

This represents the typical temperature dependence of a metal which is sketched in the following figure.



This temperature dependence is also known as Mathiessen's rule. It includes the residual resistance at zero temperature due to defects and the increase of the resistivity with temperature due to the scattering with phonons. We have also indicated the transition to a superconducting state at T_c, where the resistance vanishes. This will be discussed in a later

section.

Frequency Dependence of Electrical Conduction

The frequency dependence of the electrical conductivity can be calculated within the Drude model by solving the differential equation for the the velocity of the charge carriers

$$\frac{dv(t)}{dt}+\frac{1}{\tau}v(t)=\frac{q}{m}E(t) \text{ with } E(t)=E_0e^{i\omega t}$$

where the scattering rate has been included like in the DC case. By applying the Ansatz $v(t)=\tilde{v}_d e^{i\omega t}$ with complex $\tilde{v}_d$ one arrives at the equations

$$(i\omega\tilde{v}_d+\frac{1}{\tau}\tilde{v}_d)e^{i\omega t}=\frac{q}{m}E_0e^{i\omega t}$$
$$\Rightarrow \tilde{v}_d=\frac{q}{m}\frac{1}{i\omega+1/\tau}E_0=\frac{q}{m}\cdot\frac{1/\tau-i\omega}{\omega^2+(1/\tau)^2}E_0$$

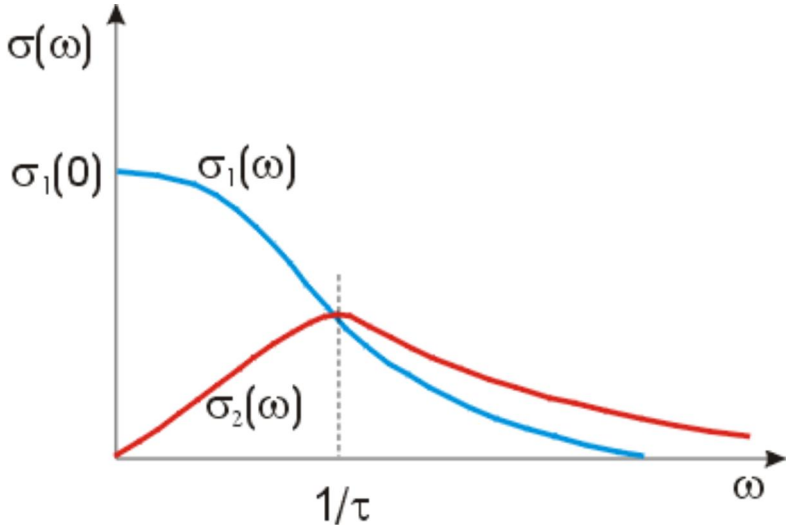
By using the definition for the frequency dependent current density and the corresponding conductivity

$$j(\omega)=nq\tilde{v}_d(\omega)E_0=\sigma(\omega)E_0$$

we derive the following expressions for the complex frequency dependent conductivity

$$\sigma(\omega)=n\frac{q^2}{m}\left(\frac{\tau-i\omega\tau^2}{1+\omega^2\tau^2}\right)=\sigma_1(\omega)-i\sigma_2(\omega)$$
$$\Rightarrow \sigma_1(\omega)=n\frac{q^2}{m}\cdot\frac{\tau}{1+\omega^2\tau^2}\ ;\ \sigma_2(\omega)=\sigma_1(\omega)\omega\tau$$
$$\Rightarrow \sigma(\omega)=\sigma_1(\omega)(1-i\omega\tau)$$

This conductivity becomes a complex quantity with a real and an imaginary contribution. At zero frequency (DC case) the imaginary part is zero, whereas the real part starts at its maximum value at zero frequency and decays to zero at infinite fervency as is sketched in the following figure.



Note that the real part of the conductivity reaches its half value at the scattering rate and crosses there the imaginary part which decreases more slowly with increasing frequency. Measuring the complex frequency dependence over a wide frequency range therefore allows to determine the scattering rate.

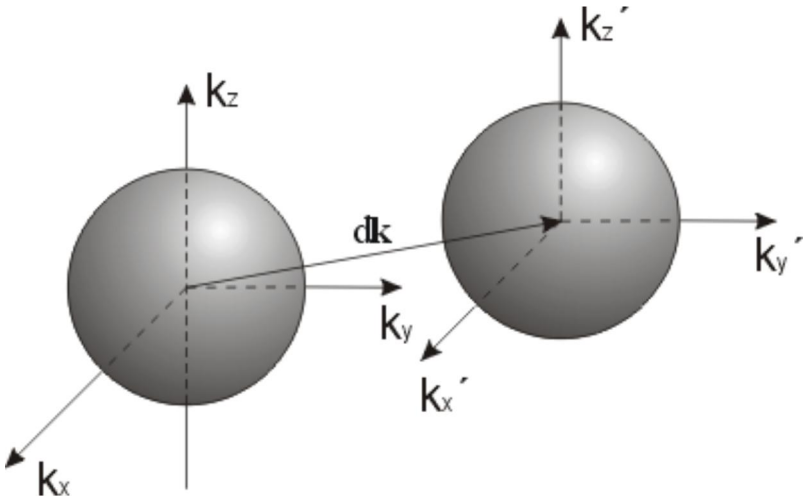
The Sommerfeld Model of Conductivity

When deriving the Drude conductivity of metals we noted that the derivation is inappropriate for metals, because it does not take the Pauli principle and the Fermi energy into account. Sommerfeld recognized this early on and proposed a different model which does take account of the quantum mechanics and the Pauli principle. This is what we are discussion in the following. Surprisingly enough, we we obtain finally the same expression for the conductivity as before.

We have learned that the momentum of free electrons can be expressed as $mv = \hbar k$. If we apply an electric field to the conduction electrons of a metal, a force is applied which leads to an accelerated motion of the electrons according to

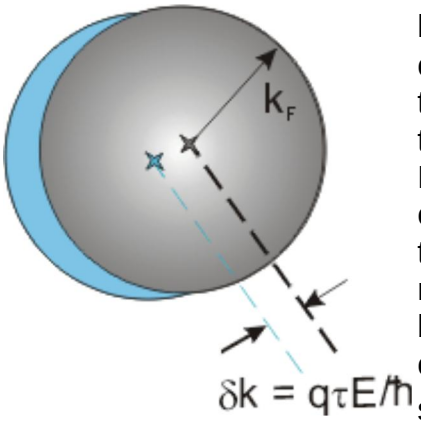
$$F = qE = \hbar \frac{dk}{dt} \Rightarrow \delta k = \frac{q}{m} E \delta t$$

which corresponds to a shift of the Fermi sphere by an amount dk during time dt. This is shown in k-space in an exaggerated way in the following figure.



If no scattering of the electrons would occur the Fermi sphere would move away. However, since scattering takes place it will be pulled back according to the relaxation equation

$$\frac{\delta k}{\delta t} = \frac{q}{\hbar}E - \frac{1}{\tau}\delta k \text{ stationary state } \Rightarrow \delta k = \frac{q}{\hbar}\tau E$$



leading to a stationary state with constantdk. The stationary shift of the Fermi sphere is directly proportional to the electric field and the scattering time of the electrons. The Pauli principle now enters the following way. Since all states of the electrons below the Fermi energy are occupied, only those electrons near the Fermi energy which find empty states can relax back. This is shown in the figure to the left. Only those electrons at the Fermi surface near the light blue shaded area(unoccupied states) can scatter backwards which drives the Fermisurface backwards, that is opposite to the force caused by the electric field. In the stationary state only the fraction $\tilde{n}/n = \delta k/k_F$ of the electrons contributes to the conduction. Following this idea of Sommerfeld leads to following current density

$$j = \tilde{n}qv_F = n\frac{\delta k}{k_F}qv_F = nq^2\tau\frac{v_F}{\hbar k_F}E \xrightarrow{\text{with } mv_F = \hbar k_F} n\frac{q^2\tau}{m}E$$

where we assume that the small fraction of scattered electrons moves with Fermi velocity. Note that finally the same conductivity as derived by the Drude model results.

Heat Conductivity of Metals

Finally we want to discuss the heat conductivity due to conduction electrons. Everybody has made the experience that a metallic spoon in a pot of hot tea or coffee feels the heat pretty quickly, much more though than with a plastic spoon. This tells us that electron provide a much better heat conductivity than phonons. Heat conductivity of electrons follows the same phenomenological kinetic equations as we encountered already in the discussion of the heat conductivity of the phonons. Starting with the definition of the heat conductivity

$$\mathbf{j_Q} = -\lambda_{th} grad T \text{ with } \lambda_{th}^{(el)} = \frac{1}{3}C_V^{(el)}v_F \cdot l$$

we need to include the parameters specific for electrons. These are the electronic heat capacity $C_V^{(el)} = \gamma T;$ with $\gamma = n\frac{\pi^2}{2}\frac{k_B}{E_F}$ and the mean free path $l = v_F\tau$, i. e. the average path length between two scattering events. Inserting also the definitions for the Fermi energy and the Fermi velocity results in

$$\lambda_{th}^{(el)} = n\frac{\pi^2}{3}\frac{k_B^2}{m}\cdot T \cdot \tau \cdot$$

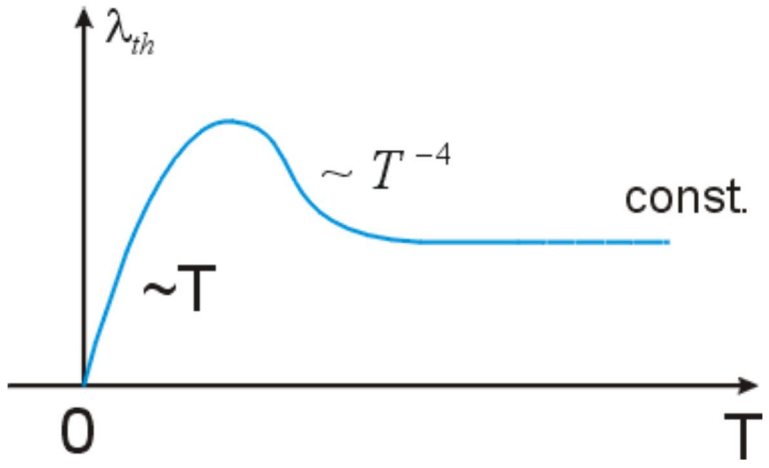
In metals the only temperature dependent parameter in the heat conductivity is the scattering time. We have discussed this already in the case of the phonon induced conductivity. Repeating these relations here results in the following equations:

$$\frac{1}{\tau} = \frac{1}{\tau_d} + \frac{1}{\tau_{ph}} \Rightarrow \tau = \frac{1}{\frac{1}{\tau_d} + \frac{1}{\tau_{ph}}}$$

$$T < T_D : \tau = \frac{1}{\frac{1}{\tau_d} + aT^5} \Rightarrow \lambda_{th}^{(el)} \sim \frac{T}{\frac{1}{\tau_d} + aT^5}$$

$$T \geq T_D : \tau = \frac{1}{\frac{1}{\tau_d} + bT} \Rightarrow \lambda_{th}^{(el)} \sim \frac{T}{\frac{1}{\tau_d} + bT} \rightarrow \frac{1}{b} = const.$$

Since the electron scattering time is determined by the local defects which are responsible for the temperature independent par and the phonons which show significantly different temperature dependence below and above the Debye temperature T_D. The following figure represents schematically the temperature dependence of the electronic heat conductivity.



At low temperature the scattering of electrons is dominated by the temperature independent defect scattering. The heat conductivity therefore grows linearly with T as expected from the kinetic equation. With increasing temperature the phonon scattering contributes a T^{-5} term which leads to a T^{-4} decay when it overcomes the defect scattering. Since well above the Debye temperature the phonons contribute a T^{-1} term which compensates the T proportional term, leading to a temperature independent heat conductivity at high temperatures.

Since the electron scattering time contributes to both, the electrical conductivity and the heat conductivity it is interesting to investigate the relation between these quantities. This was in fact first done by Wiedemann and Franz. They the relation

$$\frac{\lambda_{th}^{(el)}}{\sigma^{(el)}} = \frac{\pi^2 k_B^2}{3e^2} \cdot T = 2.45 \times 10^{-8} T \frac{W \cdot Ohm}{grad^2}$$

which is known today as the Wiedemann-Franz law. It follows right away from the relations for

these quantities which we have derived here.