Relativistic Phase Invariance of Light

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1. The Problem

We will try to show that the difference in phase of light beam between emitter and receiver is Lorentz invariant.

We will define the phase difference as

$$\phi = \frac{\Delta L}{\lambda} \tag{1}$$

where ΔL is the length of the path and λ is the wavelength of the light. Clearly we will have to define ΔL differently from the proper length, which is obviously zero. We can do this by choosing an affine parameter so the length is defined $\Delta L^2 \propto \Delta t^2 + \Delta x^2 = k(\Delta t^2 + \Delta x^2)$. We are working in units where $c = 1$.

2. Possible Solution

Consider a light beam travelling from points P and Q with

$$P = \langle t_0,\ x_0,\ 0,\ 0\rangle^T, \quad Q = \langle t_0 + dl,\ x_0 \pm dl,\ 0,\ 0\rangle^T \tag{2}$$

dl is a positive constant. The $\pm$ sign distinguishes the cases of motion in the x and $-x$ directions. The distance between P and Q is

$$\Delta L^2 = (Q - P)^T \cdot (Q - P) = 2k\ dl^2 \tag{3}$$

We will now transform P, Q with a boost in the x-direction using the transformation

$$\Lambda = \begin{bmatrix} \gamma & \pm\beta\gamma & 0 & 0 \\ \pm\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

which gives (the $\pm$ sign will be taken as positive in what follows)

$$P^{'} = \Lambda P = \left[\begin{array}{cccc} x_0\,\beta\,\gamma + t_0\,\gamma & t_0\,\beta\,\gamma + x_0\,\gamma & 0 & 0 \end{array}\right]^T \tag{5}$$

$$Q^{'} = \Lambda Q = \left[\begin{array}{cccc} (x_0 + dl)\ \beta\,\gamma + (t_0 + dl)\ \gamma & (t_0 + dl)\ \beta\,\gamma + (x_0 + dl)\ \gamma & 0 & 0 \end{array}\right]^T \tag{6}$$

and the length

$$\Delta L'^2 = (Q' - P')^T \cdot (Q' - P') = 2k\ dl^2\gamma^2\,(1 + \beta)^2 \tag{7}$$

The wavelength λ transforms with the Doppler shift formula

$$\lambda^{'} = \lambda\gamma\,(1 + \beta) = \lambda\sqrt{\frac{1 + \beta}{1 - \beta}} \tag{8}$$

and so

$$\phi = \frac{\Delta L}{\lambda} = \frac{\Delta L^{'}}{\lambda^{'}} = \phi^{'} \tag{9}$$

Performing the calculation for motion in the $-x$ direction reverses the sign of β in the transformed distance and the transformed wavelength and yields the same result.

3. The Affine Parameterisation

A parameter s is affine if $dx^{\mu}/ds = constant$. If we choose $s = x$ then this condition is clearly satisfied with $dt/dx = 1,\ dx/dx = 1,\ dy/dx = 0,\ dz/dx = 0$.

The calculations have been repeated with this parameter. This means the lengths are given by

$$\Delta L^2 = ((Q - P)[1])^2 \tag{10}$$

$$\Delta L'^2 = \left((Q' - P')[1]\right)^2 \tag{11}$$

and the only difference with the lengths in (3) and (7) is that the factor of 2 is gone, and the result remains the same.

This is to be expected because for the null direction $\Delta x = \Delta t$ so $s = 2x$ in the first calculation.

4. Conclusion

The choice of affine parameter makes no difference to the result because the constant factor generated by the choice will cancel from the final equation. This leads to the conclusion that the phase difference, as defined in (1) is a Lorentz scalar. The results are invariant under spatial rotation and coordinate translation, so it is also a Poincare invariant.