Exercises for Theoretical Physics II: Quantum Mechanics I
Sheet 6

The solutions will be discussed in the following dates:

- Group 1 (Pikovski) — Monday 5.5.14
- Group 2 (Costa) — Monday 5.05.14
- Group 3 (Steinacker) — Friday 9.05.14
- Group 4 (Dehnadi) — Wednesday 7.05.14
- Group 5 (Zauner) — Monday 5.05.14

Problem 17: Gaussian wave-packets Realistic particles are typically described by wave-packets, which have some built-in uncertainty both in position and momentum. An important example is the Gaussian wave-packet,

$$\psi_0(x) = \langle x|\psi_0\rangle = \frac{1}{(\pi\sigma^2)^{1/4}} \exp\Big(-\frac{(x-x_0)^2}{2\sigma^2}\Big) \tag{1}$$

a) verify that the above wavefunction is correctly normalized.

b) compute the expectation value $E(X) = \langle\psi_0|X|\psi_0\rangle$ and the uncertainty $(\Delta X)^2 = \langle\psi_0|(X - E(X))^2|\psi_0\rangle$ of the position operator X in this state.

c) Find the momentum representation $\tilde{\psi}_0(p) = \langle p|\psi_0\rangle$ of the wavefunction

$$\psi_0(x) = \frac{1}{(2\pi\hbar)^{1/2}} \int dp\ e^{i\frac{px}{\hbar}}\,\tilde{\psi}_0(p),$$
$$\tilde{\psi}_0(p) = \frac{1}{(2\pi\hbar)^{1/2}} \int dx\ e^{-i\frac{px}{\hbar}}\,\psi_0(x),$$

using Fourier transformation. Use this to compute the expectation value and the variance of the momentum operator $P = -i\hbar\frac{\partial}{\partial x}$. Verify the uncertainty relation.

Hint: use the formula

$$\int_{-\infty}^{\infty} dx e^{-ax^2-bx} = \sqrt{\frac{\pi}{a}}\ e^{\frac{b^2}{4a}}$$

and note that the integral over $\mathbb{R}$ of an odd function vanishes.

Problem 18: propagating Gaussian wave-packets

a) Find the time-dependent wave-packet $\psi(t,x)$ for $\psi(0,x) = \psi_0(x)$ given by the wave-packet in problem 17, by solving the Schrödinger equation with the Hamiltonian

$$H = \frac{P^2}{2m} \tag{2}$$

To do this, compute the time evolution first in the momentum representation, and then obtain the position representation.

Determine the probability density $\rho(t,x) = |\psi(t,x)|^2$, and verify that the total probability is conserved.

Compute the expectation value of the position operator X and its uncertainty at time t, and also for the momentum operator. Verify again the uncertainty relation.

b) Now assume that the initial wave-packet at time $t = 0$ is given by

$$\psi_0(x) = \frac{1}{(\pi\sigma^2)^{1/4}} \exp\left(i\frac{px}{\hbar} - \frac{(x-x_0)^2}{2\sigma^2}\right) \tag{3}$$

Obtain the wavefunction $\psi(t,x)$ at a later time $t > 0$, and compute the expectation value of X. Interpret the result.

Problem 19: We assume that the wavefunction of an electron in one dimension has the form

$$\psi(x) = c\ \chi_a(x) \tag{4}$$

where

$$\chi_a(x) = \begin{cases} 1, & x \in [-a,a] \\ 0, & \text{otherwise} \end{cases}$$

(this may happen e.g. after passing through a narrow slit).

a) Determine c

b) In this state, the energy $\frac{P^2}{2m}$ of the electron is measured. Find the probability (density) that the electron is found to have energy in the interval $[E, E+dE]$.