Problem Statement. A solid sphere of uniform density rotating at angular velocity ω abruptly separates into two hemispheres along a plane containing the axis of rotation.

Approach 1.

Angular momentum is conserved throughout the process. Let $x(t)$ be the separation between the centers of the hemispheres and $\ell(t)$ the distance between the CM of each hemisphere. Furthermore, let $\omega(t)$ denote the angular velocity of each hemisphere relative to a reference frame which rotates with the axis passing through the CM of each hemisphere, $\theta(t)$ the angular position of each hemisphere (defined so that $\theta(0) = 0$), $\Omega(t)$ the angular velocity of this rotating reference frame, and $I(t)$ the moment of inertia of the entire system.

By the parallel-axis theorem,

$$I(t) = I(0) + Mx(t)^2 = \tfrac{2}{5}MR^2 + \tfrac{1}{4}Mx(t)^2.$$

By the Pythagorean theorem,

$$\ell(t) = \sqrt{x(t)^2 + (\tfrac{3}{4}R)^2} \Leftrightarrow x(t) = \sqrt{\ell(t)^2 - (\tfrac{3}{4}R)^2}.$$

Differentiating the latter with respect to time gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\ell(t)}{\sqrt{\ell(t)^2 - (\frac{3}{4}R)^2}}\frac{\mathrm{d}\ell}{\mathrm{d}t}.$$

There is a centrifugal force

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ell(t) = \ell(t)\Omega(t)^2.$$

We also conserve angular momentum in the lab frame. Here the angular velocity of both hemispheres is $\omega(t) + \Omega(t)$, so the angular momentum due to rotation is

$$L_{\mathrm{rot}}(\tfrac{2}{5}MR^2 + \tfrac{1}{4}Mx(t)^2)(\omega(t) + \Omega(t)).$$

The translational velocity of each hemisphere relative to the CM is $-\frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}$, so the angular momentum due to relative motion is

$$L_{\mathrm{rel}} = 2(\tfrac{1}{2}M)(\tfrac{3}{8}R)(-\tfrac{1}{2}\tfrac{\mathrm{d}x}{\mathrm{d}t}) = -\tfrac{3}{16}MR\tfrac{\mathrm{d}x}{\mathrm{d}t}.$$

Angular momentum conservation then gives

$$L_i = L_{\mathrm{rot}} + L_{\mathrm{rel}} \Rightarrow \tfrac{2}{5}MR^2\omega_0 = (\tfrac{2}{5}MR^2 + \tfrac{1}{4}Mx(t)^2)(\omega(t) + \Omega(t)) - \tfrac{3}{16}MR\tfrac{\mathrm{d}x}{\mathrm{d}t}$$

$$\Rightarrow \Omega(t) = \frac{\frac{2}{5}MR^2\omega_0 + \frac{3}{16}MR\frac{\mathrm{d}x}{\mathrm{d}t}}{\frac{2}{5}MR^2 + \frac{1}{4}Mx(t)^2} - \omega(t).$$

The angle $\theta(t)$ is related to the separation distance $x(t)$ by

$$\theta(t) = \tan^{-1}\frac{x(t)}{\frac{3}{4}R}.$$

This can be differentiated to give

$$\omega(t) = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{1 + (x(t)/\frac{3}{4}R)^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x(t)}{\frac{3}{4}R}\right] = \frac{\frac{3}{4}R}{\frac{3}{4}R^2 + x(t)^2}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\frac{3}{4}R}{\ell(t)^2}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

We can use this to obtain

$$\Omega(t) = \frac{\frac{2}{5}R^2\omega_0 + \frac{3}{16}R\frac{\mathrm{d}x}{\mathrm{d}t}}{\frac{2}{5}R^2 + \frac{1}{4}x(t)^2} - \omega(t) = \frac{\frac{2}{5}R^2\omega_0 + \frac{3}{16}R\frac{\mathrm{d}x}{\mathrm{d}t}}{\frac{2}{5}R^2 + \frac{1}{4}x(t)^2} - \frac{\frac{3}{4}R}{\ell(t)^2}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\frac{2}{5}R^2\omega_0 + \frac{3}{16}R\frac{\mathrm{d}x}{\mathrm{d}t}}{\ell(t)^2 + \frac{83}{320}R^2} - \frac{\frac{3}{4}R}{\ell(t)\sqrt{\ell(t)^2 - (\frac{3}{4}R)^2}}\frac{\mathrm{d}\ell}{\mathrm{d}t}$$

$$= \frac{\frac{2}{5}R^2\omega_0}{\ell(t)^2 + \frac{83}{320}R^2} + \frac{\frac{3}{16}R\ell(t)\frac{\mathrm{d}\ell}{\mathrm{d}t}}{(\ell(t)^2 + \frac{83}{320}R^2)\sqrt{\ell(t)^2 - (\frac{3}{4}R)^2}} - \frac{\frac{3}{4}R}{\ell(t)\sqrt{\ell(t)^2 - (\frac{3}{4}R)^2}}\frac{\mathrm{d}\ell}{\mathrm{d}t}$$

$$= \frac{\frac{2}{5}R^2\omega_0}{\ell(t)^2 + \frac{83}{320}R^2} + \frac{\frac{3}{16}R\ell(t)^2\frac{\mathrm{d}\ell}{\mathrm{d}t}}{\ell(t)(\ell(t)^2 + \frac{83}{320}R^2)\sqrt{\ell(t)^2 - (\frac{3}{4}R)^2}} - \frac{\frac{3}{4}R(\ell(t)^2 + \frac{83}{320}R^2)}{\ell(t)(\ell(t)^2 + \frac{83}{320}R^2)\sqrt{\ell(t)^2 - (\frac{3}{4}R)^2}}\frac{\mathrm{d}\ell}{\mathrm{d}t}$$

$$= \frac{\frac{2}{5}R^2\omega_0}{\ell(t)^2 + \frac{83}{320}R^2} - \frac{\frac{3}{4}R(\frac{83}{320}R^2 + \frac{3}{4}\ell(t)^2)}{\ell(t)(\ell(t)^2 + \frac{83}{320}R^2)\sqrt{\ell(t)^2 - (\frac{3}{4}R)^2}}\frac{\mathrm{d}\ell}{\mathrm{d}t}$$

which gives the differential equation

$$\frac{\mathrm{d}^2\ell}{\mathrm{d}t^2} = \ell(t)\left(\frac{\frac{2}{5}R^2\omega_0}{\ell(t)^2 + \frac{83}{320}R^2} - \frac{\frac{3}{4}R(\frac{83}{320}R^2 + \frac{3}{4}\ell(t)^2)}{\ell(t)(\ell(t)^2 + \frac{83}{320}R^2)\sqrt{\ell(t)^2 - (\frac{3}{4}R)^2}}\frac{\mathrm{d}\ell}{\mathrm{d}t}\right)^2.$$

The initial conditions for this equation are $\ell(0) = \frac{3}{4}R$ and $\dot{\ell}(0) = 0$. Note that the hemispheres separate when $x = 2R$, or equivalently

$$\ell = \sqrt{(2R)^2 + (\tfrac{3}{4}R)^2} = \sqrt{4 + \tfrac{9}{16}}R = \tfrac{\sqrt{73}}{4}R.$$

If t_0 denotes the time of separation and $v = \dot{\ell}(t_0)$, then

$$\ddot{\ell}(t_0) = \frac{\sqrt{73}}{4}R\left(\frac{\frac{2}{5}R^2\omega_0}{\frac{1543}{320}R^2} - \frac{\frac{3}{4}R(\frac{589}{160}R^2)}{(\frac{\sqrt{73}}{4}R)(\frac{1543}{320}R^2)(2R)}v\right)^2$$

$$= \frac{\sqrt{73}}{4}\left(\frac{128R\omega_0}{1543} - \frac{1767}{1543\sqrt{73}}v\right)^2$$

Approach 2.

Let ω_0 be the initial angular velocity of the sphere and $\omega(r)$ the angular velocity when the centers of the hemispheres are separated by a distance $2r$, $0 < r < R$. Also assume that the system is rotating counterclockwise.

The distance between the CM of the upper hemisphere and the axis of symmetry is $\ell = \sqrt{(\frac{3}{8}R)^2 + r^2}$; the angle θ between the line joining these two points and the x-axis satisfies $\sin\theta = \frac{3}{8}R/\ell$ and $\cos\theta = r/\ell$.

We orient our coordinate system so the x-axis is parallel to the flat surface of each hemispheres and the y-axis is perpendicular to the surface.

The linear velocity of the CM of the upper hemisphere due to rotation satisfies $|\vec{v}^{\mathrm{rot}}| = \ell\omega$. This velocity vector is an angle θ counterclockwise from the positive y-axis, so

$$v_x^{\mathrm{rot}} = -|\vec{v}^{\mathrm{rot}}|\sin\theta = -(\ell\sin\theta)\omega = -\tfrac{3}{8}R\omega, \qquad v_y^{\mathrm{rot}} = |\vec{v}^{\mathrm{rot}}|\cos\theta = (\ell\cos\theta)\omega = r\omega.$$

The velocity of the upper hemisphere relative to the axis of symmetry is $v_x^{\mathrm{rel}} = \frac{\mathrm{d}x}{\mathrm{d}t}$, so in the lab frame.

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} - \tfrac{3}{8}R\omega, \qquad v_y = r\omega.$$

Both kinetic energy and angular momentum are conserved. The initial kinetic energy is $K(0) = \frac{1}{2}(\frac{2}{5}MR^2)\omega_0^2$ and the initial angular momentum is $J(0) = \frac{2}{5}MR^2\omega_0$. The final kinetic energy is given by the sum of the translational kinetic energy of the CM and the rotational kinetic energy of a hemisphere about its CM; likewise for the final angular momentum. Thus,

$$\begin{aligned}\tfrac{2}{5}MR^2\omega_0 &= I_0\omega(r) + (M\vec{r}\times\vec{v})_z = I_0\omega + M[(r, \tfrac{3}{8}R, 0)\times(\tfrac{\mathrm{d}r}{\mathrm{d}t} - \tfrac{3}{8}R\omega, r\omega)]_z \\ &= I_0\omega + M(r^2\omega - \tfrac{3}{8}R(\tfrac{\mathrm{d}r}{\mathrm{d}t} - \tfrac{3}{8}R\omega)) = \{I_0 + M[r^2 + (\tfrac{3}{8}R)^2]\}\omega - \tfrac{3}{8}MR\tfrac{\mathrm{d}r}{\mathrm{d}t}.\end{aligned}$$

We recognize the expression in braces as the moment of inertia of the hemisphere about the symmetry axis, by the parallel-axis theorem. Solving for ω gives

$$\tfrac{2}{5}MR^2\omega_0 + \tfrac{3}{8}MR\tfrac{\mathrm{d}r}{\mathrm{d}t} = (\tfrac{2}{5}MR^2 + Mr^2)\omega \Rightarrow \omega = \frac{\frac{2}{5}R^2\omega_0 + \frac{3}{8}R\frac{\mathrm{d}r}{\mathrm{d}t}}{\frac{2}{5}R^2 + r^2}.$$

Kinetic energy conservation gives

$$\begin{aligned}\tfrac{2}{5}MR^2\omega_0^2 &= I_0\omega(r)^2 + Mv^2 = I_0\omega^2 + M[(\tfrac{\mathrm{d}r}{\mathrm{d}t} - \tfrac{3}{8}R\omega)^2 + r^2\omega^2] \\ &= I_0\omega^2 + M\{[(\tfrac{\mathrm{d}r}{\mathrm{d}t})^2 - \tfrac{3}{4}R\omega\tfrac{\mathrm{d}r}{\mathrm{d}t}] + [(\tfrac{3}{8}R)^2 + r^2]\omega^2\} = I(r)\omega^2 + M[(\tfrac{\mathrm{d}r}{\mathrm{d}t})^2 - \tfrac{3}{4}R\omega\tfrac{\mathrm{d}r}{\mathrm{d}t}]\end{aligned}$$

$$\Rightarrow \frac{2}{5}MR^2\omega_0^2 = \left(\frac{2}{5}MR^2 + Mr^2\right)\left(\frac{\frac{2}{5}R^2\omega_0 + \frac{3}{8}R\frac{\mathrm{d}r}{\mathrm{d}t}}{\frac{2}{5}R^2 + r^2}\right)^2 + M\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{3}{4}MR\left(\frac{\frac{2}{5}R^2\omega_0 + \frac{3}{8}R\frac{\mathrm{d}r}{\mathrm{d}t}}{\frac{2}{5}R^2 + r^2}\right)\frac{\mathrm{d}r}{\mathrm{d}t}$$

$$\Rightarrow \frac{2}{5}R^2\omega_0^2 = \frac{(\frac{2}{5}R^2\omega_0 + \frac{3}{8}R\frac{\mathrm{d}r}{\mathrm{d}t})^2}{\frac{2}{5}R^2 + r^2} + \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{3}{4}R\left(\frac{\frac{2}{5}R^2\omega_0 + \frac{3}{8}R\frac{\mathrm{d}r}{\mathrm{d}t}}{\frac{2}{5}R^2 + r^2}\right)\frac{\mathrm{d}r}{\mathrm{d}t}$$

$$\begin{aligned}&\Rightarrow (\tfrac{2}{5}R^2 + r^2)\tfrac{2}{5}R^2\omega_0^2 = (\tfrac{2}{5}R^2\omega_0 + \tfrac{3}{8}R\tfrac{\mathrm{d}r}{\mathrm{d}t})^2 + (\tfrac{2}{5}R^2 + r^2)(\tfrac{\mathrm{d}r}{\mathrm{d}t})^2 - \tfrac{3}{4}R(\tfrac{2}{5}R^2\omega_0 + \tfrac{3}{8}R\tfrac{\mathrm{d}r}{\mathrm{d}t})\tfrac{\mathrm{d}r}{\mathrm{d}t} \\ &= (\tfrac{2}{5}R^2\omega_0)^2 + \cancel{\tfrac{3}{4}R(\tfrac{2}{5}R^2\omega_0)\tfrac{\mathrm{d}r}{\mathrm{d}t}} + (\tfrac{3}{8}R\tfrac{\mathrm{d}r}{\mathrm{d}t})^2 + (\tfrac{2}{5}R^2 + r^2)(\tfrac{\mathrm{d}r}{\mathrm{d}t})^2 - \cancel{\tfrac{3}{4}R(\tfrac{2}{5}R^2\omega_0)\tfrac{\mathrm{d}r}{\mathrm{d}t}} - 2(\tfrac{3}{8}R\tfrac{\mathrm{d}r}{\mathrm{d}t})^2\end{aligned}$$

$$= (\tfrac{2}{5}R^2\omega_0)^2 + (\tfrac{2}{5}R^2 + r^2)(\tfrac{\mathrm{d}r}{\mathrm{d}t})^2 - (\tfrac{3}{8}R\tfrac{\mathrm{d}r}{\mathrm{d}t})^2 = (\tfrac{2}{5}R^2\omega_0)^2 + (\tfrac{2}{5} - \tfrac{9}{64})R^2 + r^2)(\tfrac{\mathrm{d}r}{\mathrm{d}t})^2$$

$$\Rightarrow \cancel{(\tfrac{2}{5}R^2\omega_0)^2} + \tfrac{2}{5}R^2r^2\omega_0^2 = \cancel{(\tfrac{2}{5}R^2\omega_0)^2} + (\tfrac{83}{320}R^2 + r^2)(\tfrac{\mathrm{d}r}{\mathrm{d}t})^2$$

$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\sqrt{\frac{2}{5}}Rr\omega_0}{\sqrt{\frac{83}{320}R^2 + r^2}}.$$

This fails to give a sensible result because the solution with the initial condition $r(0) = 0$ is $r(t) \equiv 0$. Moreover, $r(0) = \epsilon$ approaches this solution as $\epsilon \to 0$.