Evaluating $\int_0^\infty \frac{\log x\, dx}{\sqrt{x}(x^2+a^2)^2}$ using contour integration

I need your help with this integral:

$$\int_0^\infty \frac{\log x\, dx}{\sqrt{x}(x^2+a^2)^2}$$

where $a > 0$. I have tried some complex integration methods, but none seems adequate for this particular one.

Is there a specific method for this kind of integrals? What contour should I use?

(integration) (complex-analysis) (definite-integrals) (improper-integrals) (contour-integration)

edited Jun 4 at 11:28 asked Jun 4 at 11:24
guest_user
77 5

1 It makes sense to substitute $\sqrt{x} = u$ – Yuriy S Jun 4 at 11:27

1 Do you know this page: en.wikipedia.org/wiki/Methods_of_contour_integration ? – b00n heT Jun 4 at 11:31

@b00nheT I didn't, thank you. – guest_user Jun 4 at 11:32

?? Thus $\frac{\partial}{\partial(a^2)} = \frac{\partial}{2a\partial a}$

4 Answers

$$\int_0^\infty \frac{\ln(x)}{\sqrt{x}\,(x^2+a^2)^2}\,dx \overset{x\to x^{1/2}}{=} \frac{1}{4}\int_0^\infty \frac{x^{-3/4}\ln(x)}{(x+a^2)^2}\,dx = -\frac{1}{4}\,\frac{\partial}{\partial(a^2)}\int_0^\infty \frac{x^{-3/4}\ln(x)}{x+a^2}\,dx$$

$$= -\frac{1}{8|a|}\,\frac{\partial}{\partial|a|}\int_0^\infty \frac{x^{-3/4}\ln(x)}{x+a^2}\,dx = -\frac{1}{8|a|}\,\frac{\partial}{\partial|a|}\left[\lim_{\mu\to-3/4}\frac{\partial}{\partial\mu}\int_0^\infty \frac{x^\mu}{x+a^2}\,dx\right]$$

$\frac{d}{dx} a^x = a^x \ln a$

I saw the same in one variable $\frac{d}{dr}\,\frac{1}{2r}$

With the *branch-cut* $z^\mu = |z|^\mu \exp(i\arg(z)\mu)$, $0 < \arg(z) < 2\pi$, $z \neq 0$, the integral is performed along a *key-hole contour*. Namely,

$$2\pi i\,|a|^{2\mu}\exp(i\pi\mu) = \int_0^\infty \frac{x^\mu}{x+a^2}\,dx + \int_\infty^0 \frac{x^\mu\exp(2\pi\mu i)}{x+a^2}\,dx$$

$$= -\exp(i\pi\mu)\,[2i\sin(\pi\mu)]\int_0^\infty \frac{x^\mu}{x+a^2}\,dx$$

$$\Longrightarrow \int_0^\infty \frac{x^\mu}{x+a^2}\,dx = -\pi\,|a|^{2\mu}\csc(\pi\mu)$$

Plug this result in (1):

$$\int_0^\infty \frac{\ln(x)}{\sqrt{x}\,(x^2+a^2)^2}\,dx = \frac{\sqrt{2}}{16}\,\pi\,\frac{6\ln(|a|)-3\pi-4}{|a|^{7/2}}$$

$= \frac{d}{d(r^2)}$

$= \frac{1}{2r}\,\frac{d}{dr}$

edited Jun 4 at 23:12 answered Jun 4 at 22:57
Felix Marin
40.8k 6 78 92

2 I will study this. Upvoted. (+1). The page is starting to look interesting. – Marko Riedel Jun 4 at 22:59

Great answer, thank you! – guest_user Jun 5 at 13:28