THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1111: Linear Algebra
Assignment 1

Due date : Sept 28, 2011 **before** 6:30 p.m.

Where to hand-in : Assignment Box outside the lifts on the 4th floor of Run Run Shaw

Remember to write down your **Name**, **Uni. no.** and **Tutorial Group number**.

• We do not count assignment grades in your final score, but your effort on assignments will be considered in case you are marginally failed.

• You have to learn how to present solutions/proofs *logically* and *clearly*. The clarity of presentation counts in tests/examinations. That's why we read your assignment work.

• If you get difficulties, you are welcome to see the instructor, tutors or seek help from the help room. See "Information" at http://hkumath.hku.hk/course/MATH1111/index.html for availabilities.

Part I: Hand-in your solutions

1. Given two nonzero vectors $\underline{x}_1 = (a \quad b)^T, \underline{x}_2 = (c \quad d)^T \in \mathbb{R}^2$.

   (a) Suppose that $\underline{x}_1$ is not a scalar multiple of $\underline{x}_2$ (i.e. $\underline{x}_1 \neq \lambda \underline{x}_2$ for any $\lambda \in \mathbb{R}$). Show that every vector $\underline{x} \in \mathbb{R}^2$ is a linear combination of $\underline{x}_1$ and $\underline{x}_2$.

   (b) Will the conclusion remain valid if $\underline{x}_1$ is a scalar multiple of $\underline{x}_2$? Justify your answer.

2. Determine all 2×2 nonsingular matrices A that satisfy $A^3 = A$.

3. Let A be a square matrix of order n. Prove the following:

   (a) If A is invertible and $AB = 0$ for some $n \times n$ matrix B, then $B = 0$.

   (b) If A is not invertible, then we can find an $n \times n$ nonzero matrix B such that $AB = 0$.

   [Here 0 means a zero matrix of appropriate size.]

4. Let A be an $m \times n$ matrix, and B be $n \times m$. Prove that if $n < m$, then AB is not invertible.

5. Prove or disprove the following:

   (a) *For any* non-singular matrix A, *there exists* a matrix B of the same order as A such that $A^2B = A$.

   (b) *There exists* a square matrix B such that *for any* non-singular matrix A of the same order as B, $A^2B = A$.

   (c) *For any* non-singular matrix A, *there exists* a vector $\underline{x} \in \mathbb{R}^n$ where n is the order of A, such that *for any* vector $\underline{y} \in \mathbb{R}^n$, $A\underline{x} = \underline{y}$.

   (d) *For any* non-singular matrix A of order n, *for any* vector $\underline{y} \in \mathbb{R}^n$, *there exists* a vector $\underline{x} \in \mathbb{R}^n$ such that $A\underline{x} = \underline{y}$.

Part II: NOT to be Handed in

6. Find a, b and c so that the system

$$\begin{cases} x + ay + cz &= 0 \\ bx + cy - 3z &= 1 \\ ax + 2y + bz &= 5 \end{cases}$$

   has the solution $x = 3$, $y = -1$, $z = 2$.

7. In each case find an invertible matrix U such that $UA = R$ in reduced row-echleon form and express U as a product of elementary matrices.

   (a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 5 & 12 & -1 \end{pmatrix}$.

   (b) $A = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 1 \end{pmatrix}$.

8. Let M be an $n \times n$ matrix and B be $n \times r$. We define the operation

$$M[B] = B^T MB.$$

   (So $M[B]$ is an $r \times r$ matrix.)

   (a) Show that $(M[B])[C] = M[BC]$ whenever either side is well-defined.

   (b) If we have, in block matrix form,

$$M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

where A is an $r \times r$ nonsingular symmetric matrix, B is $r \times (n-r)$ and D is $(n-r) \times (n-r)$, show that

$$M = \begin{pmatrix} A & 0 \\ 0 & D - A^{-1}[B] \end{pmatrix} \left[\begin{pmatrix} I_r & A^{-1}B \\ 0 & I_{n-r} \end{pmatrix} \right]$$

where I_r and I_{n-r} denote the identity matrices of order r and $n-r$ respectively. Also 0 denotes the zero matrix of suitable dimension.

9. Let A be a symmetric $n \times n$ matrix. A is said to be *positive semi-definite* if $A[\underline{x}] \geq 0$ for all vectors $\underline{x} \in \mathbb{R}^n$. We say that A is *positive definite* if $A[\underline{x}] > 0$ for all vectors $\underline{0} \neq \underline{x} \in \mathbb{R}^n$. (We use the same notation as in the last question, i.e. $A[\underline{x}] = \underline{x}^T A \underline{x}$. Also, we identify an 1×1 matrix to a number, i.e. the 1×1 matrix (3) is regarded as the number 3.)

   Define for $1 \leq r \leq n$,

$$A_r = A \left[\begin{pmatrix} I_r \\ 0 \end{pmatrix} \right]$$

   where I_r is the $r \times r$ identity matrix and 0 is the $(n-r) \times r$ zero matrix.

   (a) Show that A_r is positive semi-definite if A is positive semi-definite.

   (b) Must A_r be positive definite if A is positive definite?

   (c) Show that A is nonsingular if A is positive definite.

   (d) Show that A^{-1} is positive definite if A is positive definite.

   (e) Show that $\det A > 0$ if A is positive definite. Is the converse true?

   (f) Let a_{ii} be the (i,i)th entry of A. Suppose A is positive definite. Show that

$$\det(A) \leq a_{11} a_{22} \cdots a_{nn}.$$